COHOMOLOGIES OF CERTAIN ORBIFOLDS

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Abstract. We study the Bott-Chern cohomology of complex orbifolds obtained as quotient of a compact complex manifold by a finite group of biholomorphisms.

Introduction

In order to investigate cohomological aspects of compact complex non-Kähler manifolds, and in particular with the aim to get results allowing to construct new examples of non-Kähler manifolds, we study the cohomology of complex orbifolds.

Namely, an *orbifold* (or *V-manifold*, as introduced by I. Satake, [22]) is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n / G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, where n is the complex dimension: in other words, local geometry of orbifolds reduces to local G-invariant geometry. A special case is provided by orbifolds of global-quotient type, namely, by orbifolds $\tilde{X} = X/G$, where X is a complex manifold and G is a finite group of biholomorphisms of X; such orbifolds have been studied, among others, by D. D. Joyce in constructing examples of compact manifolds with special holonomy, see [14, 13, 15, 16]. As proven by I. Satake, and W. L. Baily, from the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools for the study of the de Rham and Dolbeault cohomology of complex orbifolds, [22, 4, 5].

In particular, an useful tool in studying the cohomological properties of non-Kähler manifolds is provided by the *Bott-Chern cohomology*, that is, the bi-graded algebra

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} .$$

After R. Bott and S. S. Chern, [7], several authors studied the Bott-Chern cohomology in many different contexts: for example, it has been recently considered by L.-S. Tseng and S.-T. Yau in the framework of Generalized Geometry and type II String Theory, [24]. While for compact Kähler manifolds X one has that the Bott-Chern cohomology is naturally isomorphic to the Dolbeault cohomology, [9, Lemma 5.15, Remark 5.16, 5.21, Lemma 5.11], in general, for compact non-Kähler manifolds X, the natural maps $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ induced by the identity are neither injective nor surjective. One says that a compact complex manifold *satisfies the $\partial\overline{\partial}$-Lemma* if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, that is, if the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective; compact Kähler manifolds provide the main examples of complex manifolds satisfying the $\partial\overline{\partial}$-Lemma, [9, Lemma 5.11], other than motivations for their study.

In this note, we study the *Bott-Chern cohomology* of compact complex orbifolds $\tilde{X} = X/G$ of global-quotient type, (where X is a compact complex manifold and G is a finite group of biholomorphisms of X,) that is, the bi-graded $\mathbb{C}$-algebra

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}$$

where $\partial\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet+1,\bullet}\tilde{X}$ and $\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet,\bullet+1}\tilde{X}$, and $\wedge^{\bullet,\bullet}\tilde{X}$ is the bi-graded $\mathbb{C}$-vector space of *differential forms* on $\tilde{X}$, that is, of G-invariant differential forms on X. We prove the following result, see Theorem 2.1.

Theorem. *Let $\tilde{X} = X/G$ be a compact complex orbifold of complex dimension n, where X is a compact complex manifold and G is a finite group of biholomorphisms of X. For any $p, q \in \mathbb{N}$, there is a canonical isomorphism*

$$H^{p,q}_{BC}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} ,$$

where $\mathcal{D}^{p,q}\tilde{X}$ denotes the space of currents of bi-degree (p,q) on $\tilde{X}$, that is, the space of G-invariant currents of bi-degree (p,q) on X.

Furthermore, given a Hermitian metric on $\tilde{X}$ (that is, a G-invariant Hermitian metric on X), there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{A} ,$$

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where $\tilde\Delta_{BC}$ and $\tilde\Delta_A$ are the 4^{th} order self-adjoint elliptic differential operators

$$\tilde\Delta_{BC} := \left(\partial\overline\partial\right)\left(\partial\overline\partial\right)^* + \left(\partial\overline\partial\right)^*\left(\partial\overline\partial\right) + \left(\overline\partial^*\partial\right)\left(\overline\partial^*\partial\right)^* + \left(\overline\partial^*\partial\right)^*\left(\overline\partial^*\partial\right) + \overline\partial^*\overline\partial + \partial^*\partial \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde X\right)$$

and

$$\tilde\Delta_A := \partial\partial^* + \overline\partial\overline\partial^* + \left(\partial\overline\partial\right)^*\left(\partial\overline\partial\right) + \left(\partial\overline\partial\right)\left(\partial\overline\partial\right)^* + \left(\overline\partial\partial^*\right)^*\left(\overline\partial\partial^*\right) + \left(\overline\partial\partial^*\right)\left(\overline\partial\partial^*\right)^* \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde X\right) .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde X\right) \simeq H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde X\right) .$$

As regards the $\partial\overline\partial$-Lemma for complex orbifolds, by adapting a result by R. O. Wells in [25], we get the following result, see Corollary 3.2.

Corollary. *Let $\tilde Y$ and $\tilde X$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde Y \to \tilde X$ be a proper surjective morphism of complex orbifolds. If $\tilde Y$ satisfies the $\partial\overline\partial$-Lemma, then also $\tilde X$ satisfies the $\partial\overline\partial$-Lemma.*

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1. Preliminaries on orbifolds

The notion of orbifold has been introduced by I. Satake in [22], with the name of *V-manifold*, and has been studied, among many others, by W. L. Baily, [4, 5].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, referring to [17, 16, 22, 4, 5].

A *complex orbifold of complex dimension n* is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, [22, Definition 2].

By definition, an object (e.g., a *differential form*, a *Riemannian metric*, a *Hermitian metric*) *on a complex orbifold* $\tilde X$ is defined locally at $x \in \tilde X$ as a G_x-invariant object on $\mathbb{C}^n$, where $G_x \subseteq \mathrm{GL}(n;\mathbb{C})$ is such that $\tilde X$ is locally isomorphic to $\mathbb{C}^n/G_x$ at x.

Given $\tilde X$ and $\tilde Y$ complex orbifolds, a *morphism $f\colon \tilde Y \to \tilde X$ of complex orbifolds* is a morphism of complex spaces given, locally at $y \in \tilde Y$, by a map $\mathbb{C}^m/H_y \to \mathbb{C}^n/G_{f(y)}$, where $\tilde Y$ is locally isomorphic to $\mathbb{C}^m/H_y$ at y and $\tilde X$ is locally isomorphic to $\mathbb{C}^n/G_{f(y)}$ at $f(y)$.

In particular, one gets a differential complex $\left(\wedge^\bullet\tilde X,\, \mathrm{d}\right)$, and a double complex $\left(\wedge^{\bullet,\bullet}\tilde X,\, \partial,\, \overline\partial\right)$. Define the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomology groups of $\tilde X$ respectively as

$$H^\bullet_{dR}\left(\tilde X;\mathbb{C}\right) := \frac{\ker \mathrm{d}}{\mathrm{im}\,\mathrm{d}} , \qquad H^{\bullet,\bullet}_{\overline\partial}\left(\tilde X\right) := \frac{\ker\overline\partial}{\mathrm{im}\,\overline\partial} ,$$

$$H^{\bullet,\bullet}_{BC}\left(\tilde X\right) := \frac{\ker\partial\cap\ker\overline\partial}{\mathrm{im}\,\partial\overline\partial} , \qquad H^{\bullet,\bullet}_{A}\left(\tilde X\right) := \frac{\ker\partial\overline\partial}{\mathrm{im}\,\partial + \mathrm{im}\,\overline\partial} .$$

The structure of double complex of $\left(\wedge^{\bullet,\bullet}\tilde X,\, \partial,\, \overline\partial\right)$ induces naturally a spectral sequence $\left\{\left(E_r^{\bullet,\bullet},\, \mathrm{d}_r\right)\right\}_{r\in\mathbb{N}}$, called *Hodge and Frölicher spectral sequence of* $\tilde X$, such that $E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline\partial}\left(\tilde X\right)$ (see, e.g., [20, §2.4]). Hence, one has the *Frölicher inequality*, see [12, Theorem 2],

$$\sum_{p+q=k} \dim_\mathbb{C} H^{p,q}_{\overline\partial}\left(\tilde X\right) \geq \dim_\mathbb{C} H^k_{dR}\left(\tilde X;\mathbb{C}\right) ,$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde X$ of complex dimension n, one can consider the $\mathbb{R}$-linear Hodge-$*$-operator $*_g\colon \wedge^\bullet\tilde X \to \wedge^{2n-\bullet}\tilde X$, and hence the 2^{nd} order self-adjoint elliptic differential operator $\Delta := [\mathrm{d},\, \mathrm{d}^*] := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} \in \mathrm{End}\left(\wedge^\bullet\tilde X\right)$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde X$ of complex dimension n, one can consider the $\mathbb{C}$-linear Hodge-$*$-operator $*_g\colon \wedge^{\bullet_1,\bullet_2}\tilde X \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde X$, and hence the 2^{nd} order self-adjoint elliptic differential operator $\overline\square := \left[\overline\partial,\, \overline\partial^*\right] := \overline\partial\,\overline\partial^* + \overline\partial^*\,\overline\partial \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde X\right)$. Furthermore, in [18, Proposition 5], and [23, §2], the following 4^{th} order self-adjoint elliptic differential operators are defined:

$$\tilde\Delta_{BC} := \left(\partial\overline\partial\right)\left(\partial\overline\partial\right)^* + \left(\partial\overline\partial\right)^*\left(\partial\overline\partial\right) + \left(\overline\partial^*\partial\right)\left(\overline\partial^*\partial\right)^* + \left(\overline\partial^*\partial\right)^*\left(\overline\partial^*\partial\right) + \overline\partial^*\overline\partial + \partial^*\partial \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde X\right)$$

and

$$\tilde\Delta_A := \partial\partial^* + \overline\partial\overline\partial^* + \left(\partial\overline\partial\right)^*\left(\partial\overline\partial\right) + \left(\partial\overline\partial\right)\left(\partial\overline\partial\right)^* + \left(\overline\partial\partial^*\right)^*\left(\overline\partial\partial^*\right) + \left(\overline\partial\partial^*\right)\left(\overline\partial\partial^*\right)^* \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde X\right) .$$

As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension n, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\underline{\mathbb{R}}_{\tilde{X}}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $\mathcal{C}^\infty_{\tilde{X}}$, the sheaf of germs of (p,q)-forms (for $p,q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{p,q}_{\tilde{X}}$, the sheaf of germs of k-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^k_{\tilde{X}}$, the sheaf of germs of bidimension-(p,q)-currents (for $p,q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,p,q} :=: \mathcal{D}^{n-p,n-q}_{\tilde{X}}$, the sheaf of germs of dimension-k-currents (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,k} :=: \mathcal{D}^{2n-k}_{\tilde{X}}$, and the sheaf of holomorphic p-forms (for $p \in \mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a compact complex orbifold, has been proven by I. Satake, [22], and by W. L. Baily, [4].

Theorem 1.1 ([22, Theorem 1], [4, Theorem H]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \check{H}^\bullet\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right) \;.$$

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \ker\Delta \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;.$$

The isomorphism $H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq \ker\Delta$ can be seen as a consequence of a more general decomposition theorem on compact orbifolds, [4, Theorem D], which holds for 2^{nd} order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result by W. L. Baily, [5, 4], holds.

Theorem 1.2 ([5, page 807], [4, Theorem K]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \check{H}^{\bullet_2}\left(\tilde{X};\Omega^{\bullet_1}_{\tilde{X}}\right) \;.$$

Furthermore, given a Hermitian metric on X, there is a canonical isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \ker\overline{\square} \;.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;.$$

2. Bott-Chern cohomology of complex orbifolds of global-quotient type

Compact complex orbifolds of the type $\tilde{X} = X/G$, where X is a compact complex manifold and G is a finite group of biholomorphisms of X, constitute one of the simplest examples of singular manifolds: more precisely, in this section, we study the Bott-Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the G-invariant $\tilde{\Delta}_{BC}$-harmonic forms on X, Theorem 2.1.

Consider

$$\tilde{X} \;=\; X/G \;,$$

where X is a compact complex manifold and G is a finite group of biholomorphisms of X: by the Bochner linearization theorem, [6, Theorem 1], see also [21, Theorem 1.7.2], $\tilde{X}$ turns out to be an orbifold as in I. Satake's definition.

Such orbifolds of global-quotient type have been considered and studied by D. D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy G_2, [14] and [16, Chapters 11-12], and examples of compact 8-dimensional manifolds with holonomy Spin(7), [13, 15] and [16, Chapters 13-14]. See also [11, 8] for the use of orbifolds of global-quotient type to construct a compact 8-dimensional simply-connected non-formal symplectic manifold (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I. K. Babenko and I. A. Taĭmanov, [3, Problem].

Since G is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

$$\mathrm{Sing}\left(\tilde{X}\right) \;=\; \left\{x\,G \in X/G \;:\; x \in X \text{ and } g\cdot x = x \text{ for some } g \in G\setminus\{\mathrm{id}_X\}\right\} \;.$$

We provide the following result, concerning Bott-Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient type.

Theorem 2.1. *Let $\tilde{X} = X/G$ be a compact complex orbifold of complex dimension n, where X is a compact complex manifold and G is a finite group of biholomorphisms of X. For any $p,q \in \mathbb{N}$, there is a canonical isomorphism*

$$H^{p,q}_{BC}\left(\tilde{X}\right) \;\simeq\; \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right)\cap\ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} \;. \tag{1}$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there are canonical isomorphisms
$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{A} \,.$$
In particular, the Hodge-$*$-operator induces an isomorphism
$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \simeq H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right) \,.$$

Proof. We use the same argument as in the proof of [1, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of $\tilde{X}$ can be computed using either differential forms or currents, the same holds true for the Bott-Chern and the Aeppli cohomologies.

Indeed, note that, for any $p,q\in\mathbb{N}$, one has the exact sequence
$$0\to\frac{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$
$$\to\frac{\ker\left(\mathrm{d}\colon \mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}\to\frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} \,,$$
where the maps are induced by the identity. By [22, Theorem 1], one has
$$\frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} \simeq \frac{\ker\left(\mathrm{d}\colon \left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\wedge^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} \,,$$
therefore it suffices to prove that the space
$$\frac{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$
can be computed using just differential forms on $\tilde{X}$.

Firstly, we note that, since, by [5, page 807],
$$\frac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \simeq \frac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} \,,$$
one has that, if $\psi\in\wedge^{r,s}\tilde{X}$ is a $\overline{\partial}$-closed differential form, then every solution $\phi\in\mathcal{D}^{r,s-1}$ of $\overline{\partial}\phi=\psi$ is a differential form up to $\overline{\partial}$-exact terms. Indeed, since $[\psi]=0$ in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$ and hence in $\frac{\ker\overline{\partial}\cap\wedge^{r,s}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$, there is a differential form $\alpha\in\wedge^{r,s-1}\tilde{X}$ such that $\psi=\overline{\partial}\alpha$. Hence, $\phi-\alpha\in\mathcal{D}^{r,s-1}\tilde{X}$ defines a class in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}}\simeq\frac{\ker\overline{\partial}\cap\wedge^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$, and hence $\phi-\alpha$ is a differential form up to a $\overline{\partial}$-exact form, and so ϕ is.

By conjugation, if $\psi\in\wedge^{r,s}\tilde{X}$ is a ∂-closed differential form, then every solution $\phi\in\mathcal{D}^{r-1,s}$ of $\partial\phi=\psi$ is a differential form up to ∂-exact terms.

Now, let
$$\omega^{p,q} = \mathrm{d}\,\eta \mod \mathrm{im}\,\partial\overline{\partial} \in \frac{\mathrm{im}\,\mathrm{d}\cap\mathcal{D}^{p,q}X}{\mathrm{im}\,\partial\overline{\partial}} \,.$$
Decomposing $\eta=:\sum_{p,q}\eta^{p,q}$ in pure-type components, where $\eta^{p,q}\in\mathcal{D}^{p,q}\tilde{X}$, the previous equality is equivalent to the system
$$\left\{\begin{array}{rcccll}
 & & \partial\eta^{p+q-1,0} & = & 0 & \mod \mathrm{im}\,\partial\overline{\partial} \\
\overline{\partial}\eta^{p+q-\ell,\ell-1} & + & \partial\eta^{p+q-\ell-1,\ell} & = & 0 & \mod \mathrm{im}\,\partial\overline{\partial} \quad \text{for} \quad \ell\in\{1,\ldots,q-1\} \\
\overline{\partial}\eta^{p,q-1} & + & \partial\eta^{p-1,q} & = & \omega^{p,q} & \mod \mathrm{im}\,\partial\overline{\partial} \\
\overline{\partial}\eta^{\ell,p+q-\ell-1} & + & \partial\eta^{\ell-1,p+q-\ell} & = & 0 & \mod \mathrm{im}\,\partial\overline{\partial} \quad \text{for} \quad \ell\in\{1,\ldots,p-1\} \\
\overline{\partial}\eta^{0,p+q-1} & & & = & 0 & \mod \mathrm{im}\,\partial\overline{\partial}
\end{array}\right. .$$
By the above argument, we may suppose that, for $\ell\in\{0,\ldots,p-1\}$, the currents $\eta^{\ell,p+q-\ell-1}$ are differential form: indeed, they are differential form up to $\overline{\partial}$-exact terms, but $\overline{\partial}$-exact terms give no contribution in the system, which is modulo $\mathrm{im}\,\partial\overline{\partial}$. Analogously, we may suppose that, for $\ell\in\{0,\ldots,q-1\}$, the currents $\eta^{p+q-\ell-1,\ell}$ are differential form. Then we may suppose that $\omega^{p,q}=\overline{\partial}\eta^{p,q-1}+\partial\eta^{p-1,q}$ is a differential form. Hence (1) is proven.

Now, we prove that, fixed a G-invariant Hermitian metric on $\tilde{X}$, the Bott-Chern cohomology of $\tilde{X}$ is isomorphic to the space of $\tilde{\Delta}_{BC}$-harmonic G-invariant forms on X. Indeed, since the elements of G commute with ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$, and hence with $\tilde{\Delta}_{BC}$, the following decomposition, [23, Théorème 2.2],
$$\wedge^{\bullet,\bullet}X = \ker\tilde{\Delta}_{BC}\oplus\partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}X\oplus\left(\partial^*\wedge^{\bullet+1,\bullet}X+\overline{\partial}^*\wedge^{\bullet,\bullet+1}X\right)$$

induces a decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_{BC}\oplus\partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}\tilde{X}\oplus\left(\partial^*\wedge^{\bullet+1,\bullet}\tilde{X}+\overline{\partial}^*\wedge^{\bullet,\bullet+1}\tilde{X}\right) \;;$$

more precisely, let $\alpha\in\wedge^{\bullet,\bullet}\tilde{X}$, that is, α is a G-invariant form on X; if α has a decomposition $\alpha = h_\alpha+\partial\overline{\partial}\beta+\left(\partial^*\gamma+\overline{\partial}^*\eta\right)$ with $h_\alpha,\beta,\gamma,\eta\in\wedge^{\bullet,\bullet}X$ such that $\tilde{\Delta}_{BC}h_\alpha=0$, then one has

$$\begin{aligned}\alpha \;=\; \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\alpha \;&=\; \left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha\right)+\partial\overline{\partial}\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\beta\right)\\ &\quad+\left(\partial^*\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\gamma\right)+\overline{\partial}^*\left(\eta\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\right)\right)\;,\end{aligned}$$

where $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha$, $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\beta$, $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\gamma$, $\eta\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\in\wedge^{\bullet,\bullet}\tilde{X}$ and

$$\tilde{\Delta}_{BC}\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha\right) \;=\; \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\left(\tilde{\Delta}_{BC}h_\alpha\right) \;=\; 0\,.$$

As regards the Aeppli cohomology, one has the decomposition, [23, §2.c],

$$\wedge^{\bullet,\bullet}X \;=\; \ker\tilde{\Delta}_A\oplus\left(\partial\wedge^{\bullet-1,\bullet}X+\overline{\partial}\wedge^{\bullet,\bullet-1}X\right)\oplus\left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}X\;,$$

and hence the decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_A\oplus\left(\partial\wedge^{\bullet-1,\bullet}\tilde{X}+\overline{\partial}\wedge^{\bullet,\bullet-1}\tilde{X}\right)\oplus\left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}\tilde{X}\;,$$

from which one gets the isomorphism $H^{\bullet,\bullet}_A\left(\tilde{X}\right)\simeq\ker\tilde{\Delta}_A$.

Finally, note that the Hodge-$*$-operator $*\colon\wedge^{\bullet_1,\bullet_2}\tilde{X}\to\wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$ sends $\tilde{\Delta}_{BC}$-harmonic forms to $\tilde{\Delta}_A$-harmonic forms, and hence it induces an isomorphism

$$*\colon H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right)\stackrel{\simeq}{\to}H^{n-\bullet_2,n-\bullet_1}_A\left(\tilde{X}\right)\;,$$

concluding the proof. □

Remark 2.2. We note that another proof of the isomorphism

$$H^{p,q}_{BC}\left(\tilde{X}\right)\;\simeq\;\frac{\ker\left(\partial\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q}\tilde{X}\right)\cap\ker\left(\overline{\partial}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}\;,$$

and a proof of the isomorphism

$$H^{p,q}_{A}\left(\tilde{X}\right)\;\simeq\;\frac{\ker\left(\partial\overline{\partial}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon\mathcal{D}^{p-1,q}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)+\operatorname{im}\left(\overline{\partial}\colon\mathcal{D}^{p,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$

follow from the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [10, §V I.12.1] and M. Schweitzer, [23, §4], see also [19, §3.2].

We recall that, for any $p,q\in\mathbb{N}$, the complex $\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\,\mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\,\mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\;:\qquad \mathcal{A}^{0,0}_{\tilde{X}}\stackrel{\mathrm{pr}\circ\mathrm{d}}{\to}\bigoplus_{\substack{r+s=1\\ r<p,\,s<q}}\mathcal{A}^{r,s}_{\tilde{X}}\to\cdots\stackrel{\mathrm{pr}\circ\mathrm{d}}{\to}\bigoplus_{\substack{r+s=p+q-2\\ r<p,\,s<q}}\mathcal{A}^{r,s}_{\tilde{X}}\stackrel{\partial\overline{\partial}}{\to}\bigoplus_{\substack{r+s=p+q\\ r\geq p,\,s\geq q}}\mathcal{A}^{r,s}_{\tilde{X}}\stackrel{\mathrm{d}}{\to}\bigoplus_{\substack{r+s=p+q\\ r\geq p,\,s\geq q}}\mathcal{A}^{r,s}_{\tilde{X}}\to\cdots\;,$$

and the complex $\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\,\mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\,\mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)\;:\qquad \mathcal{D}^{0,0}_{\tilde{X}}\stackrel{\mathrm{pr}\circ\mathrm{d}}{\to}\bigoplus_{\substack{r+s=1\\ r<p,\,s<q}}\mathcal{D}^{r,s}_{\tilde{X}}\to\cdots\stackrel{\mathrm{pr}\circ\mathrm{d}}{\to}\bigoplus_{\substack{r+s=p+q-2\\ r<p,\,s<q}}\mathcal{D}^{r,s}_{\tilde{X}}\stackrel{\partial\overline{\partial}}{\to}\bigoplus_{\substack{r+s=p+q\\ r\geq p,\,s\geq q}}\mathcal{D}^{r,s}_{\tilde{X}}\stackrel{\mathrm{d}}{\to}\bigoplus_{\substack{r+s=p+q\\ r\geq p,\,s\geq q}}\mathcal{D}^{r,s}_{\tilde{X}}\to\cdots\;,$$

where pr denotes the projection onto the appropriate space.

Take ϕ a germ of a d-closed k-form on $\tilde{X}$, with $k\in\mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant k-form on X; by the Poincaré lemma, see, e.g., [10, I.1.22], there exists ψ a germ of a $(k-1)$-form on X such that $\phi=\mathrm{d}\,\psi$; since ϕ is G-invariant, one has

$$\phi \;=\; \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\phi \;=\; \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\left(\mathrm{d}\,\psi\right) \;=\; \mathrm{d}\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\psi\right)\;,$$

that is, taking the germ of the G-invariant $(k-1)$-form

$$\tilde{\psi} \;:=\; \frac{1}{\operatorname{ord} G} \sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(k-1)$-form on $\tilde{X}$ such that $\phi = \mathrm{d}\,\tilde{\psi}$. As regards the case $k=0$, one has straightforwardly that every (G-invariant) d-closed function on X is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents, see, e.g., [10, Theorem I.2.24].

Analogously, take ϕ a germ of a $\overline{\partial}$-closed (p,q)-form (respectively, bidimension-(p,q)-current) on $\tilde{X}$, with $q \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant (p,q)-form (respectively, bidimension-(p,q)-current) on X; by the Dolbeault and Grothendieck lemma, see, e.g., [10, I.3.29], there exists ψ a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on X such that $\phi = \overline{\partial}\psi$; since ϕ is G-invariant, one has

$$\phi \;=\; \frac{1}{\operatorname{ord} G} \sum_{g\in G} g^*\phi \;=\; \frac{1}{\operatorname{ord} G} \sum_{g\in G} g^*\left(\overline{\partial}\psi\right) \;=\; \overline{\partial}\left(\frac{1}{\operatorname{ord} G} \sum_{g\in G} g^*\psi\right) \;,$$

that is, taking the germ of the G-invariant $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current)

$$\tilde{\psi} \;:=\; \frac{1}{\operatorname{ord} G} \sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on $\tilde{X}$ such that $\phi = \overline{\partial}\tilde{\psi}$. As regards the case $q=0$, one has that every (G-invariant) $\overline{\partial}$-closed bidimension-$(p,0)$-current on X is locally a holomorphic p-form, see, e.g., [10, I.3.29].

By the Poincaré lemma and the Dolbeault and Grothendieck lemma, one gets M. Schweitzer's lemma [23, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick; this allows to prove that the map

$$\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right) \to \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)$$

of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [10, §IV.12.6], for every $\ell \in \mathbb{N}$,

$$\mathbb{H}^\ell\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \mathbb{H}^\ell\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;.$$

Since, for every $k \in \mathbb{N}$, the sheaves $\mathcal{L}^k_{\tilde{X}\,p,q}$ and $\mathcal{M}^k_{\tilde{X}\,p,q}$ are fine (indeed, they are sheaves of $\left(\mathcal{C}^\infty_{\tilde{X}} \otimes_{\mathbb{R}} \mathbb{C}\right)$-modules over a paracompact space), one has, see, e.g., [10, IV.4.19, (IV.12.9)],

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\colon\ \wedge^{p,q}\tilde{X} \to \wedge^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon\ \wedge^{p,q}\tilde{X} \to \wedge^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon\ \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} \;,$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon\ \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon\ \wedge^{p-2,q-1}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon\ \wedge^{p-1,q-2}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-2,q-1}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p-1,q-2}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right)} \;,$$

proving the stated isomorphisms.

3. Complex orbifolds satisfying the $\partial\overline{\partial}$-Lemma

We recall that a bounded double complex $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ of vector spaces, whose associated simple complex is $\left(K^\bullet,\, \mathrm{d}\right)$ with $\mathrm{d} := \mathrm{d}' + \mathrm{d}''$, is said *to satisfy the* $\mathrm{d}'\,\mathrm{d}''$*-Lemma*, [9], if

$$\ker \mathrm{d}' \cap \ker \mathrm{d}'' \cap \operatorname{im} \mathrm{d} = \operatorname{im} \mathrm{d}'\,\mathrm{d}'' \;;$$

other equivalent conditions are provided in [9, Lemma 5.15].

An orbifold $\tilde{X}$ is said *to satisfy the* $\partial\overline{\partial}$*-Lemma* if the double complex $\left(\wedge^{\bullet,\bullet}\tilde{X},\, \partial,\, \overline{\partial}\right)$ satisfies the $\partial\overline{\partial}$-Lemma, that is, if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, namely, in other words, if the natural map $H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^\bullet_{dR}\left(\tilde{X};\mathbb{C}\right)$ induced by the identity is injective.

Characterizations of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma in terms of their cohomological properties have been provided by P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan in [9, Proposition 5.17, 5.21], and by the author and A. Tomassini in [2, Theorem B]. As a corollary of their characterization, P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan proved that, given X and Y compact complex manifolds of the same dimension and $f\colon X \to Y$ a holomorphic birational map, if X satisfies the $\partial\overline{\partial}$-Lemma, then also Y satisfies the $\partial\overline{\partial}$-Lemma, [9, Theorem 5.22].

In this section, we extend [9, Theorem 5.22] to the case of orbifolds, by straightforwardly adapting a result by R. O. Wells, [25, Theorem 3.1], to the orbifold case.

Theorem 3.1 (see [25, Theorem 3.1])**.** *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. Then the map $\epsilon\colon \tilde{Y} \to \tilde{X}$ induces injective maps*

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right) , \qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right) , \qquad \textit{and} \qquad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) .$$

Proof. We follow closely the proof of [25, Theorem 3.1] and adapt it to the orbifold case.

`Step 1` – *Notations.* The morphism $\epsilon\colon \tilde{Y} \to \tilde{X}$ of complex orbifolds induces morphisms

$$\epsilon^*\colon \wedge^\bullet \tilde{X} \to \wedge^\bullet \tilde{Y} \qquad \text{and} \qquad \epsilon^*\colon \wedge^{\bullet,\bullet} \tilde{X} \to \wedge^{\bullet,\bullet}\tilde{Y}$$

of $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces, and hence, by duality,

$$\epsilon_*\colon \mathcal{D}_\bullet\tilde{Y} \to \mathcal{D}_\bullet\tilde{X} \qquad \text{and} \qquad \epsilon_*\colon \mathcal{D}_{\bullet,\bullet}\tilde{Y} \to \mathcal{D}_{\bullet,\bullet}\tilde{X} .$$

Moreover, recall that, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, there are natural inclusions

$$T_\cdot\colon \wedge^\bullet X \to \mathcal{D}^\bullet X := \mathcal{D}_{2n-\bullet} X \qquad \text{and} \qquad T_\cdot\colon \wedge^{\bullet,\bullet} X \to \mathcal{D}^{\bullet,\bullet} X := \mathcal{D}_{n-\bullet,n-\bullet} X ,$$

where n is the complex dimension of X.

Both ϵ^* and ϵ_* commute with d, ∂ and $\overline{\partial}$, and hence they induce morphisms of complexes

$$\epsilon^*_{dR}\colon \left(\wedge^\bullet\tilde{X},\, \mathrm{d}\right) \to \left(\wedge^\bullet\tilde{Y},\, \mathrm{d}\right) \qquad \text{and} \qquad \epsilon^{dR}_*\colon \left(\mathcal{D}^\bullet\tilde{Y},\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet\tilde{X},\, \mathrm{d}\right) ,$$

and, for any $p \in \mathbb{N}$,

$$\epsilon^*_{\overline{\partial}}\colon \left(\wedge^{p,\bullet}\tilde{X},\, \overline{\partial}\right) \to \left(\wedge^{p,\bullet}\tilde{Y},\, \overline{\partial}\right) \qquad \text{and} \qquad \epsilon^{\overline{\partial}}_*\colon \left(\mathcal{D}^{p,\bullet}\tilde{Y},\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet}\tilde{X},\, \overline{\partial}\right) ,$$

and, for any $p, q \in \mathbb{N}$,

$$\epsilon^*_{BC}\colon \left(\wedge^{p-1,q-1}\tilde{X} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}\tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}\tilde{X} \oplus \wedge^{p,q+1}\tilde{X}\right) \to \left(\wedge^{p-1,q-1}\tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}\tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}\tilde{Y} \oplus \wedge^{p,q+1}\tilde{Y}\right)$$

and

$$\epsilon^{BC}_*\colon \left(\mathcal{D}^{p-1,q-1}\tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{Y} \oplus \wedge^{p,q+1}\tilde{Y}\right) \to \left(\mathcal{D}^{p-1,q-1}\tilde{X} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{X} \oplus \mathcal{D}^{p,q+1}\tilde{X}\right) ;$$

hence, they induce morphisms between the corresponding cohomologies:

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right) , \quad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right) , \quad \text{and} \quad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) .$$

Recall that $T_\cdot$ commutes with d, ∂ and $\overline{\partial}$, and hence it induces, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, morphisms

$$T_\cdot\colon \left(\wedge^\bullet X,\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet X,\, \mathrm{d}\right) ,$$

and, for any $p \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p,\bullet} X,\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet} X,\, \overline{\partial}\right) ,$$

and, for any $p, q \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p-1,q-1}X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}X \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}X \oplus \wedge^{p,q+1}X\right) \to \left(\mathcal{D}^{p-1,q-1}X \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}X \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}X \oplus \wedge^{p,q+1}X\right) ;$$

by [22, Theorem 1], by [5, page 807], and by Theorem 2.1, these maps are in fact quasi-isomorphisms.

`Step 3` – *It holds $\epsilon_*\, T_\cdot\, \epsilon^* = \mu \cdot T_\cdot$ for some $\mu \in \mathbb{N} \setminus \{0\}$.* Indeed, consider the diagrams

$$\begin{array}{ccc} \wedge^\bullet\tilde{Y} & \stackrel{T_\cdot}{\longrightarrow} & \mathcal{D}^\bullet\tilde{Y} \\ \epsilon^*\uparrow & & \downarrow\epsilon_* \\ \wedge^\bullet\tilde{X} & \underset{T_\cdot}{\longrightarrow} & \mathcal{D}^\bullet\tilde{X} \end{array} , \qquad \text{respectively} \qquad \begin{array}{ccc} \wedge^{\bullet,\bullet}\tilde{Y} & \stackrel{T_\cdot}{\longrightarrow} & \mathcal{D}^{\bullet,\bullet}\tilde{Y} \\ \epsilon^*\uparrow & & \downarrow\epsilon_* \\ \wedge^{\bullet,\bullet}\tilde{X} & \underset{T_\cdot}{\longrightarrow} & \mathcal{D}^{\bullet,\bullet}\tilde{X} \end{array} .$$

One has that there exists a proper analytic subset $S_{\tilde{Y}}$ of $\tilde{Y} \setminus \mathrm{Sing}\left(\tilde{Y}\right)$ such that $S_{\tilde{Y}}$ has measure zero in $\tilde{Y}$ and

$$\epsilon\lfloor_{\tilde{Y}\setminus\left(\mathrm{Sing}\left(\tilde{Y}\right)\cup S_{\tilde{Y}}\right)}\colon \tilde{Y} \setminus \left(\mathrm{Sing}\left(\tilde{Y}\right) \cup S_{\tilde{Y}}\right) \to \tilde{X} \setminus \left(\mathrm{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$$

is a finitely-sheeted covering mapping of sheeting number $\mu \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{U} :=: \{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open covering of $\tilde{X} \setminus \left(\mathrm{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$, and let $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ be an associated partition of unity. For every $\varphi, \psi \in \wedge^{\bullet,\bullet}\tilde{X}$, one has that

$$\begin{aligned}
\langle \epsilon_* T.\, \epsilon^*\varphi,\, \psi\rangle &= \langle T.\, \epsilon^*\varphi,\, \epsilon^*\psi\rangle = \int_{\tilde{Y}} \epsilon^*\varphi \wedge \epsilon^*\psi = \int_{\tilde{Y}} \epsilon^*\left(\varphi\wedge\psi\right) = \int_{\tilde{Y}-\left(\mathrm{Sing}\left(\tilde{Y}\right)\cup S_{\tilde{Y}}\right)} \epsilon^*\left(\varphi\wedge\psi\right) \\
&= \sum_{\alpha\in\mathcal{A}} \int_{\pi^{-1}(U_\alpha)} \epsilon^*\left(\rho_\alpha\left(\varphi\wedge\psi\right)\right) = \sum_{\alpha\in\mathcal{A}} \sum_{\sharp\{U\in\mathcal{U}\,:\,\pi^{-1}(U)=\pi^{-1}(U_\alpha)\}} \int_{U_\alpha} \rho_\alpha\left(\varphi\wedge\psi\right) \\
&= \mu\cdot \int_{\tilde{X}-\left(\mathrm{Sing}\left(\tilde{X}\right)\cup\epsilon\left(S_{\tilde{Y}}\right)\right)} \varphi\wedge\psi = \mu\cdot\int_{\tilde{X}}\varphi\wedge\psi = \langle \mu\, T.\varphi,\, \psi\rangle\ ,
\end{aligned}$$

and hence one gets that

$$\epsilon_*\, T.\, \epsilon^* \;=\; \mu\cdot T.\ .$$

`Step 4` – *Conclusion.* Hence one has the diagrams

$$\begin{array}{ccc}
\dfrac{\ker\left(\mathrm{d}\colon \wedge^\bullet\tilde{X}\to\wedge^{\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \wedge^{\bullet-1}\tilde{X}\to\wedge^{\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\mathrm{d}\colon \mathcal{D}^\bullet\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{dR}} & & \Big\downarrow{\scriptstyle \epsilon_*^{dR}} \\
\dfrac{\ker\left(\mathrm{d}\colon \wedge^\bullet\tilde{Y}\to\wedge^{\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\mathrm{d}\colon \wedge^{\bullet-1}\tilde{Y}\to\wedge^{\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\mathrm{d}\colon \mathcal{D}^\bullet\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)}
\end{array}\ ,$$

such that

$$\epsilon_*^{dR}\, T.\, \epsilon^*_{dR} \;=\; \mu\cdot T.\ ,$$

and

$$\begin{array}{ccc}
\dfrac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{\overline{\partial}}} & & \Big\downarrow{\scriptstyle \epsilon_*^{\overline{\partial}}} \\
\dfrac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)}
\end{array}\ ,$$

such that

$$\epsilon_*^{\overline{\partial}}\, T.\, \epsilon^*_{\overline{\partial}} \;=\; \mu\cdot T.\ ,$$

and

$$\begin{array}{ccc}
\dfrac{\ker\left(\partial\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet+1,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\colon \wedge^{\bullet-1,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)+\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\colon \mathcal{D}^{\bullet-1,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)+\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{BC}} & & \Big\downarrow{\scriptstyle \epsilon_*^{BC}} \\
\dfrac{\ker\left(\partial\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\partial\colon \wedge^{\bullet-1,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)+\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\partial\colon \mathcal{D}^{\bullet-1,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)+\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)}
\end{array}\ ,$$

such that

$$\epsilon_*^{BC}\, T.\, \epsilon^*_{BC} \;=\; \mu\cdot T.\ .$$

Since $T.$ are isomorphisms in cohomology, one gets that

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right)\to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right)\ ,\qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right)\to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right)\ ,\qquad \text{and}\qquad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\ .$$

are injective. □

Corollary 3.2. *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same dimension, and let $\epsilon\colon \tilde{Y}\to\tilde{X}$ be a proper surjective morphism of complex orbifolds. If $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-Lemma, then also $\tilde{X}$ satisfies the $\partial\overline{\partial}$-Lemma.*

Proof. One has the commutative diagram

$$\begin{array}{ccc}
H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) & \xrightarrow[1:1]{\epsilon^*_{BC}} & H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \\
\Big\downarrow{\scriptstyle \mathrm{id}^*_{\tilde{X}}} & & {\scriptstyle 1:1}\Big\downarrow{\scriptstyle \mathrm{id}^*_{\tilde{Y}}} \\
H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) & \xrightarrow[1:1]{\epsilon^*_{dR}} & H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)
\end{array}$$

where $\mathrm{id}^*_{\tilde{X}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^\bullet_{dR}\left(\tilde{X};\mathbb{C}\right)$ and $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\to H^\bullet_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are the natural maps induced in cohomology by the identity. Since $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\to H^\bullet_{dR}\left(\tilde{Y};\mathbb{C}\right)$ is injective by the assumption that $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-Lemma, and $\epsilon_{BC}^{*}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)$ and $\epsilon_{dR}^{*}\colon H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)\to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are injective by Theorem 3.1, we get that also $\mathrm{id}_{\tilde{X}}^{*}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ is injective, and hence $\tilde{X}$ satisfies the $\partial\overline{\partial}$-Lemma. □

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