COHOMOLOGIES OF CERTAIN ORBIFOLDS

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Abstract. We study the Bott-Chern cohomology of complex orbifolds obtained as quotient of a compact complex manifold by a finite group of biholomorphisms.

Introduction

In order to investigate cohomological aspects of compact complex non-Kähler manifolds, and in particular with the aim to get results allowing to construct new examples of non-Kähler manifolds, we study the cohomology of complex orbifolds.

Namely, an orbifold (or V-manifold, as introduced by I. Satake, [22]) is a singular complex space whose singularities are locally isomorphic to quotient singularities \( \mathbb{C}^n/G \), for finite subgroups \( G \subset \text{GL}(n; \mathbb{C}) \), where \( n \) is the complex dimension: in other words, local geometry of orbifolds reduces to local \( G \)-invariant geometry. A special case is provided by orbifolds of global-quotient type, namely, by orbifolds \( \tilde{X} = X/G \), where \( X \) is a complex manifold and \( G \) is a finite group of biholomorphisms of \( X \); such orbifolds have been studied, among others, by D. D. Joyce in constructing examples of compact manifolds with special holonomy, see [14, Lemma 5.16, Remark 5.16, 5.21, Lemma 5.11]. As proven by I. Satake, and W. L. Baily, from the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools for the study of the de Rham and Dolbeault cohomology of complex orbifolds, [22, 4, 5].

In particular, an useful tool in studying the cohomological properties of non-Kähler manifolds is provided by the Bott-Chern cohomology, that is, the bi-graded algebra

\[ H_{BC}^{p,q}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im} \partial \partial^c} . \]

After R. Bott and S. S. Chern, [7], several authors studied the Bott-Chern cohomology in many different contexts: for example, it has been recently considered by L.-S. Tseng and S.-T. Yau in the framework of Generalized Geometry, [24]. While for compact Kähler manifolds \( X \) one has that the Bott-Chern cohomology is naturally isomorphic to the Dolbeault cohomology, [9, Lemma 5.15, Remark 5.16, 5.21, Lemma 5.11], in general, for compact non-Kähler manifolds \( X \), the natural maps \( H_{BC}^{p,q}(X) \to H_{\partial}^{p,q}(X) \) and \( H_{BC}^{p,q}(X) \to H_{\partial\partial}^{p,q}(X; \mathbb{C}) \) induced by the identity are neither injective nor surjective. One says that a compact complex manifold satisfies the \( \partial\partial \)-Lemma if every \( \partial \)-closed \( \partial\partial \)-exact form is \( \partial\partial \)-exact, that is, if the natural map \( H_{BC}^{p,q}(X) \to H_{\partial\partial}^{p,q}(X; \mathbb{C}) \) is injective; compact Kähler manifolds provide the main examples of complex manifolds satisfying the \( \partial\partial \)-Lemma, [9, Lemma 5.11], other than motivations for their study.

In this note, we study the Bott-Chern cohomology of compact complex orbifolds \( \tilde{X} = X/G \) of global-quotient type, (where \( X \) is a compact complex manifold and \( G \) is a finite group of biholomorphisms of \( X \)), that is, the bi-graded \( \mathbb{C} \)-algebra

\[ H_{BC}^{p,q}(\tilde{X}) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im} \partial \partial^c} , \]

where \( \partial : \wedge^{\bullet,\bullet} \tilde{X} \to \wedge^{\bullet+1,\bullet} \tilde{X} \) and \( \bar{\partial} : \wedge^{\bullet,\bullet} \tilde{X} \to \wedge^{\bullet,\bullet+1} \tilde{X} \), and \( \wedge^{\bullet,\bullet} \tilde{X} \) is the bi-graded \( \mathbb{C} \)-vector space of differential forms on \( \tilde{X} \), that is, of \( G \)-invariant differential forms on \( X \). We prove the following result, see Theorem 3.1.

**Theorem.** Let \( \tilde{X} = X/G \) be a compact complex orbifold of complex dimension \( n \), where \( X \) is a compact complex manifold and \( G \) is a finite group of biholomorphisms of \( X \). For any \( p, q \in \mathbb{N} \), there is a canonical isomorphism

\[ H_{BC}^{p,q}(\tilde{X}) \simeq \frac{\ker (\partial : D^p,q \tilde{X} \to D^{p+1,q} \tilde{X}) \cap \ker (\bar{\partial} : D^p,q \tilde{X} \to D^{p,q+1} \tilde{X})}{\text{im}(\partial \partial^c : D^{p-1,q-1} \tilde{X} \to D^{p,q} \tilde{X})} , \]

where \( D^p,q \tilde{X} \) denotes the space of currents of bi-degree \( (p,q) \) on \( \tilde{X} \), that is, the space of \( G \)-invariant currents of bi-degree \( (p,q) \) on \( X \).

Furthermore, given a Hermitian metric on \( \tilde{X} \) (that is, a \( G \)-invariant Hermitian metric on \( X \)), there are canonical isomorphisms

\[ H_{BC}^{p,q}(\tilde{X}) \simeq \ker \Delta_{BC} \quad \text{and} \quad H_{A}^{p,q}(\tilde{X}) \simeq \ker \Delta_{A} , \]

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where $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_{A}$ are the $4^{th}$ order self-adjoint elliptic differential operators

$$\tilde{\Delta}_{BC} := (\tilde{\partial} \tilde{\partial})^* + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + \tilde{\partial} \tilde{\partial} + \partial \partial \in \text{End} \left( \wedge^{\bullet} \tilde{X} \right)$$

and

$$\tilde{\Delta}_{A} := \partial \partial^* + \tilde{\partial} \tilde{\partial}^* + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial}) (\tilde{\partial} \tilde{\partial})^* + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* \in \text{End} \left( \wedge^{\bullet} \tilde{X} \right).$$

In particular, the Hodge-$s$-operator induces an isomorphism

$$H^{\bullet, \bullet}_{BC} (\tilde{X}) \simeq H^{n-n}_{A} (\tilde{X}).$$

As regards the $\tilde{\partial} \tilde{\partial}$-Lemma for complex orbifolds, by adapting a result by R. O. Wells in [25], we get the following result, see Corollary 3.2.

**Corollary.** Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon : \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. If $\tilde{Y}$ satisfies the $\tilde{\partial} \tilde{\partial}$-Lemma, then also $\tilde{X}$ satisfies the $\tilde{\partial} \tilde{\partial}$-Lemma.

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## 1. Preliminaries on orbifolds

The notion of orbifold has been introduced by I. Satake in [22], with the name of $V$-manifold, and has been studied, among many others, by W. L. Baily, [4, 5].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, referring to [17, 16, 22, 4, 5].

A complex orbifold of complex dimension $n$ is a singular complex space whose singularities are locally isomorphic to quotient singularities $C^n/G$, for finite subgroups $G \subset GL(n; C)$, [22, Definition 2].

By definition, an object (e.g., a differential form, a Riemannian metric, a Hermitian metric) on a complex orbifold $\tilde{X}$ is defined locally at $x \in \tilde{X}$ as a $G_x$-invariant object on $C^n$, where $G_x \subseteq GL(n; C)$ is such that $\tilde{X}$ is locally isomorphic to $C^n / G_x$ at $x$.

Given $\tilde{X}$ and $\tilde{Y}$ complex orbifolds, a morphism $f : \tilde{Y} \to \tilde{X}$ of complex orbifolds is a morphism of complex spaces given, locally at $y \in \tilde{Y}$, by a map $C^m / H_y \to C^n / G_{f(y)}$, where $\tilde{Y}$ is locally isomorphic to $C^m / H_y$ at $y$ and $\tilde{X}$ is locally isomorphic to $C^n / G_{f(y)}$ at $f(y)$.

In particular, one gets a differential complex $\left( \wedge^{\bullet} \tilde{X}, d \right)$, and a double complex $\left( \wedge^{\bullet, \bullet} \tilde{X}, \partial, \tilde{\partial} \right)$. Define the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomology groups of $\tilde{X}$ respectively as

$$H_{dR}^\bullet (\tilde{X}; C) := \frac{\ker d}{\im d}, \quad H^{\bullet, \bullet}_{\tilde{\partial}} (\tilde{X}) := \frac{\ker \tilde{\partial}}{\im \tilde{\partial}},$$

$$H^{\bullet, \bullet}_{BC} (\tilde{X}) := \frac{\ker \partial \cap \ker \tilde{\partial}}{\im \partial \tilde{\partial}}, \quad H^{\bullet, \bullet}_{A} (\tilde{X}) := \frac{\ker \partial \tilde{\partial}}{\im \partial + \im \tilde{\partial}}.$$

The structure of double complex of $\left( \wedge^{\bullet, \bullet} \tilde{X}, \partial, \tilde{\partial} \right)$ induces naturally a spectral sequence $\{ (E_r^{p,q}, d_r) \}_{r \in \mathbb{N}}$, called Hodge and Frölicher spectral sequence of $\tilde{X}$, such that $E_1^{p,q} \simeq H^{p,q}_{\tilde{\partial}} (\tilde{X})$ (see, e.g., [20, §2.4]). Hence, one has the Frölicher inequality, see [12, Theorem 2],

$$\sum_{p+q=k} \dim_C H^{p,q}_{\tilde{\partial}} (\tilde{X}) \geq \dim_C H^k_{dR} (\tilde{X}; C),$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde{X}$ of complex dimension $n$, one can consider the $\mathbb{R}$-linear Hodge+$s$-operator $\ast : \wedge^{\bullet} \tilde{X} \to \wedge^{2n-\bullet} \tilde{X}$, and hence the $2^{nd}$ order self-adjoint elliptic differential operator $\Delta := [d, d^*] := d d^* + d^* d \in \text{End} \left( \wedge^{\bullet} \tilde{X} \right)$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde{X}$ of complex dimension $n$, one can consider the $\mathbb{C}$-linear Hodge+$s$-operator $\ast : \wedge^{\bullet, \bullet} \tilde{X} \to \wedge^{n-n} \tilde{X}$, and hence the $2^{nd}$ order self-adjoint elliptic differential operator $\Box := [\tilde{\partial}, \tilde{\partial}^*] := \tilde{\partial} \tilde{\partial} + \tilde{\partial} \tilde{\partial} \in \text{End} \left( \wedge^{\bullet, \bullet} \tilde{X} \right)$. Furthermore, in [18, Proposition 5], and [23, §2], the following $4^{th}$ order self-adjoint elliptic differential operators are defined:

$$\tilde{\Delta}_{BC} := (\tilde{\partial} \tilde{\partial}) (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* \in \text{End} \left( \wedge^{\bullet} \tilde{X} \right)$$

and

$$\tilde{\Delta}_{A} := \partial \partial^* + \tilde{\partial} \tilde{\partial}^* + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial}) (\tilde{\partial} \tilde{\partial})^* + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial}) + (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* (\tilde{\partial} \tilde{\partial})^* \in \text{End} \left( \wedge^{\bullet} \tilde{X} \right).$$
As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension $n$, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\mathbb{R}\tilde{X}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $C^\infty_{\tilde{X}}$, the sheaf of germs of $(p,q)$-forms (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{p,q}_{\tilde{X}}$, the sheaf of germs of $k$-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^k_{\tilde{X}}$, the sheaf of germs of bidimension-$(p,q)$-currents (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X},p,q} := \mathcal{D}^{n-p,n-q}_{\tilde{X}}$, the sheaf of germs of dimension-$k$-currents (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X},k} := \mathcal{D}^{n-k}_{\tilde{X}}$, and the sheaf of holomorphic $p$-forms (for $p \in \mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a compact complex orbifold, has been proven by I. Satake, [22], and by W. L. Baily, [4].

**Theorem 1.1** ([22, Theorem 1], [4, Theorem H]). Let $\tilde{X}$ be a compact complex orbifold of complex dimension $n$. There is a canonical isomorphism

\[ H^*_{dR}(\tilde{X}; \mathbb{R}) \simeq H^*(\tilde{X}; \mathbb{R}\tilde{X}) . \]

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

\[ H^*_{dR}(\tilde{X}; \mathbb{R}) \simeq \ker \Delta . \]

In particular, the Hodge-$*$-operator induces an isomorphism

\[ H^*_{dR}(\tilde{X}; \mathbb{R}) \simeq H^{2n-*}_{dR}(\tilde{X}; \mathbb{R}) . \]

The isomorphism $H^*_{dR}(\tilde{X}; \mathbb{R}) \simeq \ker \Delta$ can be seen as a consequence of a more general decomposition theorem on compact orbifolds, [4, Theorem D], which holds for $2^{nd}$ order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result by W. L. Baily, [5, 4], holds.

**Theorem 1.2** ([5, page 807], [4, Theorem K]). Let $\tilde{X}$ be a compact complex orbifold of complex dimension $n$. There is a canonical isomorphism

\[ H^*_{\bar{\partial}}(\tilde{X}; \mathbb{R}) \simeq H^{*,2}(\tilde{X}; \Omega^*_{\tilde{X}}) . \]

Furthermore, given a Hermitian metric on $\tilde{X}$, there is a canonical isomorphism

\[ H^*_{\bar{\partial}}(\tilde{X}) \simeq \ker \Box . \]

In particular, the Hodge-$*$-operator induces an isomorphism

\[ H^*_{\bar{\partial}}(\tilde{X}) \simeq H^{n-*;*,n-*}_{\bar{\partial}}(\tilde{X}) . \]

2. **Bott-Chern Cohomology of Complex Orbifolds of Global-Quotient Type**

Compact complex orbifolds of the type $\tilde{X} = X / G$, where $X$ is a compact complex manifold and $G$ is a finite group of biholomorphisms of $X$, constitute one of the simplest examples of singular manifolds: more precisely, in this section, we study the Bott-Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the $G$-invariant $\Delta_{BC}$-harmonic forms on $X$, Theorem 2.1.

Consider

\[ \tilde{X} = X / G , \]

where $X$ is a compact complex manifold and $G$ is a finite group of biholomorphisms of $X$: by the Bochner linearization theorem, [6, Theorem 1], see also [21, Theorem 1.7.2], $\tilde{X}$ turns out to be an orbifold as in I. Satake’s definition.

Such orbifolds of global-quotient type have been considered and studied by D. D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy $G_2$, [14] and [16, Chapters 11-12], and examples of compact 8-dimensional manifolds with holonomy Spin(7), [13, 15] and [16, Chapters 13-14]. See also [11, 8] for the use of orbifolds of global-quotient type to construct a compact 8-dimensional simply-connected non-formal symplectic manifold (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I. K. Babenko and I. A. Taimanov, [3, Problem].

Since $G$ is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

\[ \text{Sing}(\tilde{X}) = \{ x G \in X / G : x \in X \text{ and } g \cdot x = x \text{ for some } g \in G \setminus \{ \text{id}_X \} \} . \]

We provide the following result, concerning Bott-Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient type.

**Theorem 2.1.** Let $\tilde{X} = X / G$ be a compact complex orbifold of complex dimension $n$, where $X$ is a compact complex manifold and $G$ is a finite group of biholomorphisms of $X$. For any $p, q \in \mathbb{N}$, there is a canonical isomorphism

\[ H^p_{BC}(\tilde{X}) \simeq \frac{\ker \left( \partial : \mathcal{D}^{p,q}(\tilde{X}) \to \mathcal{D}^{p+1,q}(\tilde{X}) \right) \cap \ker \left( \partial_{\overline{\partial}} : \mathcal{D}^{p,q}(\tilde{X}) \to \mathcal{D}^{p,q+1}(\tilde{X}) \right)}{\text{Im} \left( \partial_{\overline{\partial}} : \mathcal{D}^{p-1,q+1}(\tilde{X}) \to \mathcal{D}^{p,q}(\tilde{X}) \right)} . \]
Furthermore, given a Hermitian metric on \( \tilde{X} \), there are canonical isomorphisms
\[
H_{BC}^{p,q}(\tilde{X}) \simeq \ker \Delta_{BC} \quad \text{and} \quad H_A^{p,q}(\tilde{X}) \simeq \ker \Delta_A .
\]

In particular, the Hodge-\( s \)-operator induces an isomorphism
\[
H_{BC}^{p,q}(\tilde{X}) \simeq H_A^{p-s-n-q}(\tilde{X}) .
\]

Proof. We use the same argument as in the proof of [1, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of \( \tilde{X} \) can be computed using either differential forms or currents, the same holds true for the Bott-Chern and the Aeppli cohomologies.

Indeed, note that, for any \( p, q \in \mathbb{N} \), one has the exact sequence
\[
0 \to \frac{\text{im} \left( \partial \left( D^{p,q-1} \tilde{X} \otimes \mathbb{C} \right) \to \left( D^{p+q} \tilde{X} \otimes \mathbb{C} \right) \right)}{\ker \left( \partial \left( D^{p,q} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+1} \tilde{X} \right)} \to \frac{\text{im} \left( \partial \left( D^{p,q} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+1} \tilde{X} \right)}{\ker \left( \partial \left( D^{p,q+1} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+2} \tilde{X} \right)},
\]
where the maps are induced by the identity. By [22, Theorem 1], one has
\[
\frac{\ker \left( \partial \left( D^{p,q} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+1} \tilde{X} \right)}{\text{im} \left( \partial \left( D^{p,q} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+1} \tilde{X} \right)} \simeq \frac{\ker \left( \partial \left( D^{p,q+1} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+2} \tilde{X} \right)}{\text{im} \left( \partial \left( D^{p,q+1} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+2} \tilde{X} \right)},
\]
therefore it suffices to prove that the space
\[
\frac{\text{im} \left( \partial \left( D^{p,q+1} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+2} \tilde{X} \right)}{\ker \left( \partial \left( D^{p,q+1} \tilde{X} \otimes \mathbb{C} \right) \to D^{p,q+2} \tilde{X} \right)}
\]
can be computed using just differential forms on \( \tilde{X} \).

Firstly, we note that, since, [5, page 807],
\[
\frac{\ker \left( \beta : D^{r,*} \tilde{X} \to D^{r,k+1} \tilde{X} \right)}{\text{im} \left( \beta : D^{r,k} \tilde{X} \to D^{r,k+1} \tilde{X} \right)} \simeq \frac{\ker \left( \beta : \wedge^{r,*} \tilde{X} \to \wedge^{r,k+1} \tilde{X} \right)}{\text{im} \left( \beta : \wedge^{r,k} \tilde{X} \to \wedge^{r,k+1} \tilde{X} \right)},
\]
one has that, if \( \psi \in \wedge^{r,*} \tilde{X} \) is a \( \beta \)-closed differential form, then every solution \( \phi \in D^{r-1,*} \) of \( \beta \phi = \psi \) is a differential form up to \( \beta \)-exact terms. Indeed, since \( [\psi] = 0 \) in \( \ker \frac{\pi \cap \beta^*}{\text{im} \beta^*} \), there is a differential form \( \alpha \in \wedge^{r-1,*} \tilde{X} \) such that \( \psi = \beta \alpha \). Hence, \( \phi - \alpha \in D^{r-1,*} \tilde{X} \) defines a class in \( \ker \frac{\beta - \beta \beta^*}{\text{im} \beta^*} \simeq \ker \frac{\pi \cap \beta^*}{\text{im} \beta^*} \), and hence \( \phi - \alpha \) is a differential form up to a \( \beta \)-exact form, and so \( \phi \) is.

By conjugation, if \( \psi \in \wedge^{r,*} \tilde{X} \) is a \( \overline{\beta} \)-closed differential form, then every solution \( \phi \in D^{r-1,*} \) of \( \overline{\beta} \phi = \psi \) is a differential form up to \( \overline{\beta} \)-exact terms.

Now, let
\[
\omega^{p,q} = d \eta \mod \text{im} \partial \beta \in \text{im} \partial \beta \cdot \text{im} \partial \beta .
\]
Decomposing \( \eta = \sum_{p,q} \eta^{p,q} \) in pure-type components, where \( \eta^{p,q} \in D^{p,q} \tilde{X} \), the previous equality is equivalent to the system
\[
\begin{aligned}
\partial \eta^{p+q-1,0} &= 0 \mod \text{im} \partial \beta \\
\overline{\beta} \eta^{p+q-\ell,\ell-1} + \partial \eta^{p+q-\ell-1,\ell} &= 0 \mod \text{im} \partial \beta \quad \text{for } \ell \in \{1, \ldots, q-1\} \\
\overline{\beta} \eta^{p,q-1} + \partial \eta^{p,q-\ell+1} &= \omega^{p,q} \mod \text{im} \partial \beta \\
\overline{\beta} \eta^{p,q-\ell+1} + \partial \eta^{p,q-\ell-1} &= 0 \mod \text{im} \partial \beta \quad \text{for } \ell \in \{1, \ldots, p-1\} \\
\overline{\beta} \eta^{0,p,q-1} &= 0 \mod \text{im} \partial \beta
\end{aligned}
\]
By the above argument, we may suppose that, for \( \ell \in \{0, \ldots, p-1\} \), the currents \( \eta^{p+q-\ell,\ell} \) are differential form: indeed, they are differential form up to \( \partial \)-exact terms, but \( \partial \)-exact terms give no contribution in the system, which is modulo \( \text{im} \partial \beta \). Analogously, we may suppose that, for \( \ell \in \{0, \ldots, q-1\} \), the currents \( \eta^{p+q-\ell-1,\ell} \) are differential form. Then we may suppose that \( \omega^{p,q} = \overline{\beta} \eta^{p,q-1} + \partial \eta^{p,q-1} \) is a differential form. Hence (1) is proven.

Now, we prove that, fixed a \( G \)-invariant Hermitian metric on \( \tilde{X} \), the Bott-Chern cohomology of \( \tilde{X} \) is isomorphic to the space of \( \Delta_{BC} \)-harmonic \( G \)-invariant forms on \( X \). Indeed, since the elements of \( G \) commute with \( \partial, \overline{\partial}, \partial^*, \) and \( \overline{\partial}^* \), and hence with \( \Delta_{BC} \), the following decomposition, [23, Théorème 2.2],
\[
\wedge^{*,*} X = \ker \Delta_{BC} \oplus \text{im} \partial \beta \wedge^{*,*,1-1} X \oplus \left( \partial^* \wedge^{*,*+1} X + \overline{\beta} \wedge^{*,*+1} X \right)
\]
induces a decomposition
\[ \wedge^n \tilde{X} = \ker \tilde{\Delta}_{BC} \oplus \partial \bar{\partial} \wedge^{n-1} \tilde{X} \oplus \left( \partial^* \wedge^{n+1} \tilde{X} + \bar{\partial}^* \wedge^n \tilde{X} \right) \]
more precisely, let \( \alpha \in \wedge^n \tilde{X} \), that is, \( \alpha \) is a \( G \)-invariant form on \( X \); if \( \alpha \) has a decomposition \( \alpha = h_\alpha + \partial \bar{\partial} \beta + \left( \partial^* \gamma + \bar{\partial}^* \eta \right) \) with \( h_\alpha, \beta, \gamma, \eta \in \wedge^n X \) such that \( \tilde{\Delta}_{BC} h_\alpha = 0 \), then one has
\[
\alpha = \frac{1}{\ord G} \sum_{g \in G} g^* \alpha = \left( \frac{1}{\ord G} \sum_{g \in G} g^* h_\alpha \right) + \partial \bar{\partial} \left( \frac{1}{\ord G} \sum_{g \in G} g^* \beta \right) + \left( \partial^* \left( \frac{1}{\ord G} \sum_{g \in G} g^* \gamma \right) + \bar{\partial}^* \left( \frac{1}{\ord G} \sum_{g \in G} g^* \right) \right),
\]
where \( \frac{1}{\ord G} \sum_{g \in G} g^* h_\alpha, \frac{1}{\ord G} \sum_{g \in G} g^* \beta, \frac{1}{\ord G} \sum_{g \in G} g^* \gamma, \frac{1}{\ord G} \sum_{g \in G} g^* \in \wedge^n \tilde{X} \) and
\[
\tilde{\Delta}_{BC} \left( \frac{1}{\ord G} \sum_{g \in G} g^* h_\alpha \right) = \frac{1}{\ord G} \sum_{g \in G} g^* \left( \tilde{\Delta}_{BC} h_\alpha \right) = 0.
\]
As regards the Aeppli cohomology, one has the decomposition, [23, §2.c],
\[
\wedge^n \tilde{X} = \ker \tilde{\Delta}_A \oplus \left( \partial \wedge^{n-1} \tilde{X} + \bar{\partial} \wedge^n \tilde{X} \right) \oplus \left( \partial \bar{\partial} \right)^* \wedge^{n+1} \tilde{X},
\]
and hence the decomposition
\[
\wedge^n \tilde{X} = \ker \tilde{\Delta}_A \oplus \left( \partial \wedge^{n-1} \tilde{X} + \bar{\partial} \wedge^n \tilde{X} \right) \oplus \left( \partial \bar{\partial} \right)^* \wedge^{n+1} \tilde{X},
\]
from which one gets the isomorphism \( H_{BC}^{n,*} \left( \tilde{X} \right) \simeq \ker \tilde{\Delta}_A \).

Finally, note that the Hodge-\( \ast \)-operator \( \ast : \wedge^{*, n} \tilde{X} \to \wedge^{n-*, n} \tilde{X} \) sends \( \tilde{\Delta}_{BC} \)-harmonic forms to \( \tilde{\Delta}_A \)-harmonic forms, and hence it induces an isomorphism
\[
\ast : H_{BC}^{n,*} \left( \tilde{X} \right) \cong H_A^{n-*, n} \left( \tilde{X} \right),
\]
concluding the proof.

\[\square\]

**Remark 2.2.** We note that another proof of the isomorphism
\[
H_{BC}^{p,q} \left( \tilde{X} \right) \simeq \frac{\ker \left( \partial : D^{p,q} \tilde{X} \to D^{p+1,q} \tilde{X} \right) \cap \ker \left( \bar{\partial} : D^{p,q} \tilde{X} \to D^{p,q+1} \tilde{X} \right)}{\im \left( \partial \bar{\partial} : D^{p-1,q+1} \tilde{X} \to D^{p,q} \tilde{X} \right)},
\]
and a proof of the isomorphism
\[
H_A^{p,q} \left( \tilde{X} \right) \simeq \frac{\ker \left( \partial \bar{\partial} : D^{p,q} \tilde{X} \to D^{p,q} \tilde{X} \right)}{\im \left( \partial : D^{p-1,q} \tilde{X} \to D^{p,q} \tilde{X} \right) + \im \left( \bar{\partial} : D^{p,q-1} \tilde{X} \to D^{p,q} \tilde{X} \right)}
\]
follow from the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [10, §V 1.12.1] and M. Schweitzer, [23, §4], see also [19, §3.2].

We recall that, for any \( p, q \in \mathbb{N} \), the complex \( \left( L_{X,p,q}^*, dL_{X,p,q}^* \right) \) of sheaves is defined as
\[
\left( L_{X,p,q}^*, dL_{X,p,q}^* \right) : \mathcal{A}_{X}^{0,0} \xrightarrow{\pr_0} \bigoplus \mathcal{A}_{X}^{r,s} \xrightarrow{\pr_0} \cdots \xrightarrow{\pr_0} \bigoplus \mathcal{A}_{X}^{r,s} \xrightarrow{\partial} \bigoplus \mathcal{A}_{X}^{r,s} \xrightarrow{\partial} \bigoplus \mathcal{A}_{X}^{r,s} \xrightarrow{d} \bigoplus \mathcal{A}_{X}^{r,s} \to \cdots,
\]
and the complex \( \left( M_{X,p,q}^*, dM_{X,p,q}^* \right) \) of sheaves is defined as
\[
\left( M_{X,p,q}^*, dM_{X,p,q}^* \right) : \mathcal{D}_{X}^{0,0} \xrightarrow{\pr_0} \bigoplus \mathcal{D}_{X}^{r,s} \xrightarrow{\pr_0} \cdots \xrightarrow{\pr_0} \bigoplus \mathcal{D}_{X}^{r,s} \xrightarrow{\partial} \bigoplus \mathcal{D}_{X}^{r,s} \xrightarrow{\partial} \bigoplus \mathcal{D}_{X}^{r,s} \xrightarrow{d} \bigoplus \mathcal{D}_{X}^{r,s} \to \cdots,
\]
where \( \pr \) denotes the projection onto the appropriate space.

Take \( \phi \) a germ of a d-closed k-form on \( \tilde{X} \), with \( k, \in \mathbb{N} \setminus \{0\} \), that is, a germ of a \( G \)-invariant k-form on \( X \); by the Poincaré lemma, see, e.g., [10, 1.1.22], there exists \( \psi \) a germ of a \( (k-1) \)-form on \( X \) such that \( \phi = d \psi \); since \( \phi \) is \( G \)-invariant, one has
\[
\phi = \frac{1}{\ord G} \sum_{g \in G} g^* \phi = \frac{1}{\ord G} \sum_{g \in G} g^* (d \psi) = d \left( \frac{1}{\ord G} \sum_{g \in G} g^* \psi \right),
\]
that is, taking the germ of the $G$-invariant $(k-1)$-form
\[ \tilde{\psi} := \frac{1}{\text{ord } G} \sum_{g \in G} g^* \psi \]
on $X$, one gets a germ of a $(k-1)$-form on $\tilde{X}$ such that $\phi = d \tilde{\psi}$. As regards the case $k = 0$, one has straightforwardly that every $(G$-invariant) $d$-closed function on $X$ is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents, see, e.g., [10, Theorem I.2.24].

Analogously, take a germ of a $\overline{\partial}$-closed $(p, q)$-form (respectively, bidimension-$(p, q)$-current) on $\tilde{X}$, with $q \in \mathbb{N} \setminus \{0\}$, that is, a germ of a $G$-invariant $(p, q)$-form (respectively, bidimension-$(p, q)$-current) on $X$; by the Dolbeaut and Grothendieck lemma, see, e.g., [10, I.3.29], there exists $\psi$ a germ of a $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current) on $X$ such that $\phi = \overline{\partial}\psi$; since $\phi$ is $G$-invariant, one has
\[ \phi = \frac{1}{\text{ord } G} \sum_{g \in G} g^* \phi = \frac{1}{\text{ord } G} \sum_{g \in G} g^* (\overline{\partial}\psi) = \overline{\partial} \left( \frac{1}{\text{ord } G} \sum_{g \in G} g^* \psi \right), \]
that is, taking the germ of the $G$-invariant $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current)
\[ \tilde{\psi} := \frac{1}{\text{ord } G} \sum_{g \in G} g^* \psi \]
on $X$, one gets a germ of a $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current) on $\tilde{X}$ such that $\phi = \overline{\partial}\tilde{\psi}$. As regards the case $q = 0$, one has that every $(G$-invariant) $\overline{\partial}$-closed bidimension-$(p, 0)$-current on $X$ is locally a holomorphic $p$-form, see, e.g., [10, I.3.29].

By the Poincaré lemma and the Dolbeaut and Grothendieck lemma, one gets M. Schweitzer’s lemma [23, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick; this allows to prove that the map
\[ (L^\cdot_{X, p, q}, d_{L^\cdot_{X, p, q}}) \to (M^\cdot_{X, p, q}, d_{M^\cdot_{X, p, q}}) \]
of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [10, §IV.12.6], for every $\ell \in \mathbb{N}$,
\[ H^{\ell} \left( \tilde{X}; (L^\cdot_{X, p, q}, d_{L^\cdot_{X, p, q}}) \right) \cong H^{\ell} \left( \tilde{X}; (M^\cdot_{X, p, q}, d_{M^\cdot_{X, p, q}}) \right). \]

Since, for every $k \in \mathbb{N}$, the sheaves $L^k_{X, p, q}$ and $M^k_{X, p, q}$ are fine (indeed, they are sheaves of $\left( \bigodot^\infty \otimes_{\mathbb{R}} \mathbb{C} \right)$-modules over a paracompact space), one has, see, e.g., [10, IV.4.19, (IV.12.9)],
\[ H^{p+q-1} \left( \tilde{X}; (L^\cdot_{X, p, q}, d_{L^\cdot_{X, p, q}}) \right) \cong \frac{\ker (\partial; \bigwedge^{p,q} \tilde{X} \to \bigwedge^{p+1,q} \tilde{X}) \cap \ker (\overline{\partial}; \bigwedge^{p+q-1} \tilde{X} \to \bigwedge^{p,q} \tilde{X})}{\text{im} (\overline{\partial}; \bigwedge^{p-1,q-1} \tilde{X} \to \bigwedge^{p,q} \tilde{X})} \]
and
\[ H^{p+q-1} \left( \tilde{X}; (M^\cdot_{X, p, q}, d_{M^\cdot_{X, p, q}}) \right) \cong \frac{\ker (\overline{\partial}; \bigwedge^{p+1,q} \tilde{X} \to \bigwedge^{p,q+1} \tilde{X}) \cap \ker (\partial; \bigwedge^{p,q} \tilde{X} \to \bigwedge^{p,q+1} \tilde{X})}{\text{im} (\partial; \bigwedge^{p-1,q-1} \tilde{X} \to \bigwedge^{p,q} \tilde{X})} \]
and
\[ H^{p+q-2} \left( \tilde{X}; (L^\cdot_{X, p, q}, d_{L^\cdot_{X, p, q}}) \right) \cong \frac{\ker (\overline{\partial}; \bigwedge^{p-1,q-1} \tilde{X} \to \bigwedge^{p,q} \tilde{X})}{\text{im} (\partial; \bigwedge^{p-2,q-1} \tilde{X} \to \bigwedge^{p-1,q-1} \tilde{X}) + \text{im} (\partial; \bigwedge^{p-1,q-2} \tilde{X} \to \bigwedge^{p-1,q-1} \tilde{X})} \]
and
\[ H^{p+q-2} \left( \tilde{X}; (M^\cdot_{X, p, q}, d_{M^\cdot_{X, p, q}}) \right) \cong \frac{\ker (\partial; \bigwedge^{p-1,q-1} \tilde{X} \to \bigwedge^{p,q} \tilde{X})}{\text{im} (\overline{\partial}; \bigwedge^{p-2,q-1} \tilde{X} \to \bigwedge^{p-1,q-1} \tilde{X}) + \text{im} (\overline{\partial}; \bigwedge^{p-1,q-2} \tilde{X} \to \bigwedge^{p-1,q-1} \tilde{X})} \]
proving the stated isomorphisms.

3. **Complex orbifolds satisfying the $\partial\overline{\partial}$-lemma**

We recall that a bounded double complex $(K^\cdot, d', d'')$ of vector spaces, whose associated simple complex is $(K^\cdot, d)$ with $d := d' + d''$, is said to satisfy the $d'd''$-Lemma, [9], if
\[ \ker d' \cap \ker d'' \cap \text{im } d = \text{im } d' \cap \text{im } d'' ; \]
other equivalent conditions are provided in [9, Lemma 5.15].

An orbifold $\tilde{X}$ is said to satisfy the $\partial\overline{\partial}$-Lemma if the double complex $(\bigwedge \cdot \cdot \cdot \tilde{X}, \partial, \overline{\partial})$ satisfies the $\partial\overline{\partial}$-Lemma, that is, if every $\partial$-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, namely, in other words, if the natural map $H^{**}_{BC} \left( \tilde{X} \right) \to H_{dR}^{**} \left( \tilde{X}; \mathbb{C} \right)$ induced by the identity is injective.
Theorem 3.1 (see [25, Theorem 3.1]). Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon: \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. Then the map $\epsilon: \tilde{Y} \to \tilde{X}$ induces injective maps

$$
\epsilon^*: H^*_{\partial} (\tilde{X}; \mathbb{R}) \to H^*_{\partial} (\tilde{Y}; \mathbb{R}) \quad \text{and} \quad \epsilon^*: H^{p,q}_{\partial} (\tilde{X}) \to H^{p,q}_{\partial} (\tilde{Y}) \quad \text{and} \quad \epsilon^*: H^{p+\cdot}_{\partial} (\tilde{X}) \to H^{p+\cdot}_{\partial} (\tilde{Y}).
$$

Proof. We follow closely the proof of [25, Theorem 3.1] and adapt it to the orbifold case.

Step 1 - Notations. The morphism $\epsilon: \tilde{Y} \to \tilde{X}$ of complex orbifolds induces morphisms

$$
\epsilon^*: \wedge^n \tilde{X} \to \wedge^n \tilde{Y} \quad \text{and} \quad \epsilon^*: \wedge^n \tilde{X} \to \wedge^n \tilde{Y},
$$

of $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces, and hence, by duality,

$$
\epsilon_*: D_\wedge \tilde{Y} \to D_\wedge \tilde{X} \quad \text{and} \quad \epsilon_*: D_{\wedge^n} \tilde{Y} \to D_{\wedge^n} \tilde{X}.
$$

Moreover, recall that, for $X \in \{\tilde{X}, \tilde{Y}\}$, there are natural inclusions

$$
T: \wedge^n X \to D^n X := D_{2n-n} X \quad \text{and} \quad T: \wedge^n X \to D^n X := D_{n-n} X,
$$

where $n$ is the complex dimension of $X$.

Both $\epsilon^*$ and $\epsilon_*$ commute with $d$, $\partial$ and $\overline{\partial}$, and hence they induce morphisms of complexes

$$
\epsilon^*_DR: (\wedge^n X, d) \to (\wedge^n Y, d) \quad \text{and} \quad \epsilon^*_DR: (D^n Y, d) \to (D^n X, d),
$$

and, for any $p \in \mathbb{N}$,

$$
\epsilon^*_\overline{\partial}R: (\wedge^p \cdot X, \overline{\partial}) \to (\wedge^p \cdot Y, \overline{\partial}) \quad \text{and} \quad \epsilon^*_\overline{\partial}R: (D^p \cdot Y, \overline{\partial}) \to (D^p \cdot X, \overline{\partial}),
$$

and, for any $p, q \in \mathbb{N}$,

$$
\epsilon^*_BC: \left(\wedge^{p-1,q-1} X \overline{\partial} \wedge^{p+\cdot} X \oplus \wedge^{p+1,q-1} X \right) \to \left(\wedge^{p-1,q-1} Y \overline{\partial} \wedge^{p+\cdot} Y \oplus \wedge^{p+1,q-1} Y \right)
$$

and

$$
\epsilon^*_BC: \left(D^{p-1,q-1} Y \overline{\partial} D^{p+\cdot} Y \oplus \wedge^{p+1,q-1} Y \right) \to \left(D^{p-1,q-1} X \overline{\partial} D^{p+\cdot} X \oplus \wedge^{p+1,q-1} X \right);
$$

hence, they induce morphisms between the corresponding cohomologies:

$$
\epsilon^*_DR: H^*_{\partial} (\tilde{X}; \mathbb{R}) \to H^*_{\partial} (\tilde{Y}; \mathbb{R}), \quad \epsilon^*_\overline{\partial}R: H^{p,\cdot}_{\partial} (\tilde{X}) \to H^{p,\cdot}_{\partial} (\tilde{Y}), \quad \text{and} \quad \epsilon^*_BC: H^{p,\cdot}_{BC} (\tilde{X}) \to H^{p,\cdot}_{BC} (\tilde{Y}).
$$

Recall that $T$ commutes with $d$, $\partial$ and $\overline{\partial}$, and hence it induces, for $X \in \{\tilde{X}, \tilde{Y}\}$, morphisms

$$
T: (\wedge^n X, d) \to (D^n X, d),
$$

and, for any $p \in \mathbb{N}$,

$$
T: (\wedge^p \cdot X, \overline{\partial}) \to (D^p \cdot X, \overline{\partial}),
$$

and, for any $p, q \in \mathbb{N}$,

$$
T: \left(\wedge^{p-1,q-1} X \overline{\partial} \wedge^{p+\cdot} X \oplus \wedge^{p+1,q-1} X \right) \to \left(D^{p-1,q-1} X \overline{\partial} D^{p+\cdot} X \oplus \wedge^{p+1,q-1} X \right);
$$

by [22, Theorem 1], by [5, page 807], and by Theorem 2.1, these maps are in fact quasi-isomorphisms.

Step 3 - It holds $\epsilon_*: T: \epsilon^* = \mu \cdot T$, for some $\mu \in \mathbb{N} \setminus \{0\}$. Indeed, consider the diagrams

$$
\begin{array}{ccc}
\wedge^n \tilde{Y} & \xrightarrow{T} & D^n \tilde{Y} \\
\epsilon_\wedge \downarrow & & \downarrow \epsilon_* \\
\wedge^n \tilde{X} & \xrightarrow{T} & D^n \tilde{X}
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
\wedge^n \tilde{Y} & \xrightarrow{T} & D^n \tilde{Y} \\
\epsilon^* \downarrow & & \downarrow \epsilon^* \\
\wedge^n \tilde{X} & \xrightarrow{T} & D^n \tilde{X}
\end{array}
$$

One has that there exists a proper analytic subset $S_{\tilde{Y}}$ of $\tilde{Y} \setminus \text{Sing}(\tilde{Y})$ such that $S_{\tilde{Y}}$ has measure zero in $\tilde{Y}$ and

$$
\epsilon_\wedge_{\text{Sing}(\tilde{Y}) \cup S_{\tilde{Y}}}: \tilde{Y} \setminus \left(\text{Sing}(\tilde{Y}) \cup S_{\tilde{Y}}\right) \to \tilde{X} \setminus \left(\text{Sing}(\tilde{X}) \cup \epsilon(S_{\tilde{Y}})\right)
$$
is a finitely-sheeted covering mapping of sheeting number \( \mu \in \mathbb{N} \setminus \{0\} \). Let \( \mathcal{U} := \{U_\alpha\}_{\alpha \in A} \) be an open covering of \( \tilde{X} \setminus \left( \text{Sing} \left( \tilde{X} \right) \cup \epsilon(S_F) \right) \), and let \( \{\rho_\alpha\}_{\alpha \in A} \) be an associated partition of unity. For every \( \varphi, \psi \in \wedge^* \tilde{X} \), one has that

\[
\langle \epsilon_* T^* \varphi, \psi \rangle = \langle T^* \epsilon^* \varphi, \epsilon^* \psi \rangle = \int_Y \epsilon^* \varphi \wedge \epsilon^* \psi = \int_Y \epsilon^* (\varphi \wedge \psi) = \int_{Y - \left( \text{Sing}(Y) \cup S_F \right)} \epsilon^* (\varphi \wedge \psi)
\]

\[
= \sum_{\alpha \in A} \int_{\pi^{-1}(U_\alpha)} \epsilon^*(\rho_\alpha (\varphi \wedge \psi)) = \sum_{\alpha \in A} \sum_{U \in \mathcal{U} : \pi^{-1}(U) = \pi^{-1}(U_\alpha)} \int_{U_\alpha} \rho_\alpha (\varphi \wedge \psi)
\]

and hence one gets that

\[
\epsilon_* T \epsilon^* = \mu \cdot T .
\]

**Step 4 - Conclusion.** Hence one has the diagrams

\[
\begin{array}{ccc}
\ker(d: \Lambda^* \tilde{X} \to \Lambda^{*+1} \tilde{X}) & \xrightarrow{\epsilon^*_\mathcal{R}} & \ker(d: \mathcal{D}^* \tilde{X} \to \mathcal{D}^{*+1} \tilde{X}) \\
\im(d: \Lambda^* \tilde{X} \to \Lambda^{*+1} \tilde{X}) & \xrightarrow{\epsilon_\mathcal{R}} & \im(d: \mathcal{D}^* \tilde{X} \to \mathcal{D}^{*+1} \tilde{X}) \\
\end{array}
\]

such that

\[
\epsilon_\mathcal{R}^* T \epsilon^*_\mathcal{R} = \mu \cdot T .
\]

and

\[
\begin{array}{ccc}
\ker(d: \Lambda^* \tilde{Y} \to \Lambda^{*+1} \tilde{Y}) & \xrightarrow{\epsilon^*_\mathcal{R}} & \ker(d: \mathcal{D}^* \tilde{Y} \to \mathcal{D}^{*+1} \tilde{Y}) \\
\im(d: \Lambda^* \tilde{Y} \to \Lambda^{*+1} \tilde{Y}) & \xrightarrow{\epsilon_\mathcal{R}} & \im(d: \mathcal{D}^* \tilde{Y} \to \mathcal{D}^{*+1} \tilde{Y}) \\
\end{array}
\]

such that

\[
\epsilon_\mathcal{R}^* T \epsilon^*_\mathcal{R} = \mu \cdot T .
\]

Since \( T \) are isomorphisms in cohomology, one gets that

\[
\epsilon^*_\mathcal{R}: H^*_\mathcal{R}(\tilde{X}; \mathbb{R}) \to H^*_\mathcal{R}(\tilde{Y}; \mathbb{R}) , \quad \epsilon^*_\mathcal{B}: H^*_\mathcal{B}(\tilde{X}) \to H^*_\mathcal{B}(\tilde{Y}) , \quad \text{and} \quad \epsilon^*_\mathcal{R}: H^*_\mathcal{R}(\tilde{X}) \to H^*_\mathcal{R}(\tilde{Y})
\]

are injective.

**Corollary 3.2.** Let \( \tilde{Y} \) and \( \tilde{X} \) be compact complex orbifolds of the same dimension, and let \( \epsilon: \tilde{Y} \to \tilde{X} \) be a proper surjective morphism of complex orbifolds. If \( \tilde{Y} \) satisfies the \( \partial \overline{\partial} \)-Lemma, then also \( \tilde{X} \) satisfies the \( \partial \overline{\partial} \)-Lemma.

**Proof.** One has the commutative diagram

\[
\begin{array}{ccc}
H^*_\mathcal{B}(\tilde{X}) & \xrightarrow{\epsilon^*_\mathcal{B}} & H^*_\mathcal{B}(\tilde{Y}) \\
\downarrow \text{id}^*_\mathcal{X} & & \downarrow \text{id}^*_\mathcal{Y} \\
H^*_\mathcal{R}(\tilde{X}; \mathbb{C}) & \xrightarrow{\epsilon^*_\mathcal{R}} & H^*_\mathcal{R}(\tilde{Y}; \mathbb{C})
\end{array}
\]

where \( \text{id}^*_\mathcal{X}: H^*_\mathcal{B}(\tilde{X}) \to H^*_\mathcal{R}(\tilde{X}; \mathbb{C}) \) and \( \text{id}^*_\mathcal{Y}: H^*_\mathcal{B}(\tilde{Y}) \to H^*_\mathcal{R}(\tilde{Y}; \mathbb{C}) \) are the natural maps induced in cohomology by the identity. Since \( \epsilon^*_\mathcal{B}: H^*_\mathcal{B}(\tilde{X}) \to H^*_\mathcal{R}(\tilde{X}; \mathbb{C}) \) is injective by the assumption that \( \tilde{Y} \) satisfies the \( \partial \overline{\partial} \)-Lemma,
and $\epsilon^*_{BC}: H^{\bullet, \bullet}_{BC}(\tilde{X}) \to H^{\bullet, \bullet}_{BC}(\tilde{Y})$ and $\epsilon^*_{dR}: H^{\bullet, \bullet}_{dR}(\tilde{X}; \mathbb{C}) \to H^{\bullet, \bullet}_{dR}(\tilde{Y}; \mathbb{C})$ are injective by Theorem 3.1, we get that also $\text{id}^*_{\tilde{X}}: H^{\bullet, \bullet}_{BC}(\tilde{X}) \to H^{\bullet, \bullet}_{dR}(\tilde{X}; \mathbb{C})$ is injective, and hence $\tilde{X}$ satisfies the $\partial \bar{\partial}$-Lemma. $\square$

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