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Cohomologies of certain orbifolds

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ABSTRACT

We study the Bott–Chern cohomology of complex orbifolds obtained as a quotient of a compact complex manifold by a finite group of biholomorphisms.

Introduction

In order to investigate cohomological aspects of compact complex non-Kähler manifolds, and in particular with the aim to get results allowing to construct new examples of non-Kähler manifolds, we study the cohomology of complex orbifolds.

Namely, an *orbifold* (or *V-manifold*, as introduced by I. Satake, [1]) is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, where n is the complex dimension: in other words, local geometry of orbifolds reduces to local G-invariant geometry. A special case is provided by orbifolds of global-quotient type, namely, by orbifolds $\tilde{X} = X/G$, where X is a complex manifold and G is a finite group of biholomorphisms of X; such orbifolds have been studied, among others, by D.D. Joyce in constructing examples of compact manifolds with special holonomy, see [2–5]. As proven by I. Satake, and W.L. Baily, from the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools for the study of the de Rham and the Dolbeault cohomology of complex orbifolds, [1,6,7].

In particular, useful tools in studying the cohomological properties of non-Kähler manifolds are provided by the *Bott–Chern cohomology*, that is, the bi-graded algebra

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}},$$

and by the *Aeppli cohomology*, that is, the bi-graded $H^{\bullet,\bullet}_{BC}(X)$-module

$$H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}}.$$

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After R. Bott and S.S. Chern, [8], and A. Aeppli, [9], several authors studied the Bott–Chern and Aeppli cohomologies in many different contexts: for example, they have been recently considered by L.-S. Tseng and S.-T. Yau in the framework of Generalized Geometry and type II String Theory, [10]. While for compact Kähler manifolds X one has that the Bott–Chern cohomology and the Aeppli cohomology are naturally isomorphic to the Dolbeault cohomology, [11, Lemma 5.15, Remarks 5.16, 5.21, Lemma 5.11], in general, for compact non-Kähler manifolds X, the natural maps $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$, and $H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X)$ and $H^{\bullet}_{dR}(X;\mathbb{C}) \to H^{\bullet,\bullet}_{A}(X)$ induced by the identity are neither injective nor surjective. One says that a compact complex manifold *satisfies the* $\partial\overline{\partial}$*-lemma* if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, that is, if the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective; compact Kähler manifolds provide the main examples of complex manifolds satisfying the $\partial\overline{\partial}$-lemma, [11, Lemma 5.11], other than motivations for their study.

In this note, we study the *Bott–Chern cohomology* and the *Aeppli cohomology* of compact complex orbifolds $\tilde{X} = X/G$ of global-quotient type (where X is a compact complex manifold and G is a finite group of biholomorphisms of X), that is,

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \quad \text{and} \quad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial + \operatorname{im}\overline{\partial}},$$

where $\partial\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet+1,\bullet}\tilde{X}$ and $\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet,\bullet+1}\tilde{X}$, and $\wedge^{\bullet,\bullet}\tilde{X}$ is the bi-graded $\mathbb{C}$-vector space of *differential forms* on $\tilde{X}$, that is, of G-invariant differential forms on X. We prove the following result in Section 2.

Theorem 1. *Let $\tilde{X} = X/G$ be a compact complex orbifold of complex dimension n, where X is a compact complex manifold and G is a finite group of biholomorphisms of X. For any $p, q \in \mathbb{N}$, there is a canonical isomorphism*

$$H^{p,q}_{BC}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}, \tag{1}$$

where $\mathcal{D}^{p,q}\tilde{X}$ denotes the space of currents of bi-degree (p,q) on $\tilde{X}$, that is, the space of G-invariant currents of bi-degree (p,q) on X.

Furthermore, given a Hermitian metric on $\tilde{X}$ (that is, a G-invariant Hermitian metric on X), there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{BC} \quad \text{and} \quad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{A},$$

where $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_{A}$ are the 4th order self-adjoint elliptic differential operators

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \in \operatorname{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right)$$

and

$$\tilde{\Delta}_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* \in \operatorname{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right).$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \simeq H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right).$$

We notice that the previous theorem in the case of compact complex manifolds has been proven by M. Schweitzer in [12].

As regards the $\partial\overline{\partial}$-lemma for complex orbifolds, by adapting a result by R.O. Wells in [13], we get the following result.

Theorem 2. *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. If $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-lemma, then also $\tilde{X}$ satisfies the $\partial\overline{\partial}$-lemma.*

1. Preliminaries on orbifolds

The notion of orbifolds has been introduced by I. Satake in [1], with the name of *V-manifold*, and has been studied, among many others, by W.L. Baily, [6,7].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, referring to [14,5,1,6,7].

A *complex orbifold of complex dimension n* is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, [1, Definition 2].

By definition, an object (e.g., a *differential form*, a *Riemannian metric*, a *Hermitian metric*) *on a complex orbifold* $\tilde{X}$ is defined locally at $x \in \tilde{X}$ as a G_x-invariant object on $\mathbb{C}^n$, where $G_x \subseteq \mathrm{GL}(n;\mathbb{C})$ is such that $\tilde{X}$ is locally isomorphic to $\mathbb{C}^n/G_x$ at x.

Given $\tilde{X}$ and $\tilde{Y}$ complex orbifolds, a *morphism* $f\colon \tilde{Y} \to \tilde{X}$ *of complex orbifolds* is a morphism of complex spaces given, locally at $y \in \tilde{Y}$, by a map $\mathbb{C}^m/H_y \to \mathbb{C}^n/G_{f(y)}$, where $\tilde{Y}$ is locally isomorphic to $\mathbb{C}^m/H_y$ at y and $\tilde{X}$ is locally isomorphic to $\mathbb{C}^n/G_{f(y)}$ at $f(y)$.

In particular, one gets a differential complex $\left(\wedge^\bullet \tilde{X},\, \mathrm{d}\right)$, and a double complex $\left(\wedge^{\bullet,\bullet} \tilde{X},\, \partial,\, \overline{\partial}\right)$. Define the de Rham, Dolbeault, Bott–Chern, and Aeppli cohomology groups of $\tilde{X}$ respectively as

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{C}\right) := \frac{\ker \mathrm{d}}{\mathrm{im}\,\mathrm{d}}, \qquad H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) := \frac{\ker\overline{\partial}}{\mathrm{im}\,\overline{\partial}},$$
$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker\partial \cap \ker\overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}}, \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) := \frac{\ker\partial\overline{\partial}}{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}}.$$

The structure of double complex of $\left(\wedge^{\bullet,\bullet} \tilde{X},\, \partial,\, \overline{\partial}\right)$ induces naturally a spectral sequence $\left\{\left(E_r^{\bullet,\bullet},\, \mathrm{d}_r\right)\right\}_{r\in\mathbb{N}}$, called *Hodge and Frölicher spectral sequence of* $\tilde{X}$, such that $E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right)$ (see, e.g., [15, Section 2.4]). Hence, one has the *Frölicher inequality*, see [16, Theorem 2],

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}\left(\tilde{X}\right) \geq \dim_{\mathbb{C}} H^k_{dR}\left(\tilde{X};\mathbb{C}\right),$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{R}$-linear Hodge-$*$-operator $*_g\colon \wedge^\bullet \tilde{X} \to \wedge^{2n-\bullet} \tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\Delta := [\mathrm{d},\, \mathrm{d}^*] := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} \in \mathrm{End}\left(\wedge^\bullet \tilde{X}\right)$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{C}$-linear Hodge-$*$-operator $*_g\colon \wedge^{\bullet_1,\bullet_2} \tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1} \tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\overline{\Box} := \left[\overline{\partial},\, \overline{\partial}^*\right] := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \in \mathrm{End}\left(\wedge^{\bullet,\bullet} \tilde{X}\right)$. Furthermore, in [17, Proposition 5], and [12, Section 2], the following 4th order self-adjoint elliptic differential operators are defined:

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \in \mathrm{End}\left(\wedge^{\bullet,\bullet} \tilde{X}\right)$$

and

$$\tilde{\Delta}_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* \in \mathrm{End}\left(\wedge^{\bullet,\bullet} \tilde{X}\right).$$

As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension n, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\underline{\mathbb{R}}_{\tilde{X}}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $\mathcal{C}^\infty_{\tilde{X}}$, the sheaf of germs of (p, q)-forms (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{p,q}_{\tilde{X}}$, the sheaf of germs of k-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^k_{\tilde{X}}$, the sheaf of germs of bidimension-(p, q)-currents (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,p,q} =: \mathcal{D}^{n-p,n-q}_{\tilde{X}}$, the sheaf of germs of dimension-k-currents (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,k} =: \mathcal{D}^{2n-k}_{\tilde{X}}$, and the sheaf of holomorphic p-forms (for $p \in \mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a compact complex orbifold, has been proven by I. Satake, [1], and by W.L. Baily, [6].

Theorem 3 ([1, Theorem 1], [6, Theorem H]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq \check{H}^\bullet\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right).$$

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq \ker\Delta.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right).$$

The isomorphism $H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \simeq \ker\Delta$ can be seen as a consequence of a more general decomposition theorem on compact orbifolds, [6, Theorem D], which holds for 2nd order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result by W.L. Baily, [7,6], holds.

Theorem 4 *([7, p. 807], [6, Theorem K]).* *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \simeq \check{H}^{\bullet_2}\left(\tilde{X};\, \Omega^{\bullet_1}_{\tilde{X}}\right).$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \simeq \ker \overline{\square}.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \simeq H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right).$$

2. Bott–Chern cohomology of complex orbifolds of global-quotient type

Compact complex orbifolds of the type $\tilde{X} = X/G$, where X is a compact complex manifold and G is a finite group of biholomorphisms of X, constitute one of the simplest examples of singular manifolds: more precisely, in this section, we study the Bott–Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the G-invariant $\tilde{\Delta}_{BC}$-harmonic forms on X, Theorem 1.

Consider

$$\tilde{X} = X/G,$$

where X is a compact complex manifold and G is a finite group of biholomorphisms of X: by the Bochner linearization theorem, [18, Theorem 1], see also [19, Theorem 1.7.2], $\tilde{X}$ turns out to be an orbifold as in I. Satake's definition.

Such orbifolds of global-quotient type have been considered and studied by D.D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy G_2, [2] and [5, Chapters 11–12], and examples of compact 8-dimensional manifolds with holonomy Spin(7), [3,4] and [5, Chapters 13–14]. See also [20,21] for the use of orbifolds of global-quotient type to construct a compact 8-dimensional simply-connected non-formal symplectic manifold (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I.K. Babenko and I.A. Taĭmanov, [22, Problem].

Since G is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

$$\operatorname{Sing}\left(\tilde{X}\right) = \left\{x\,G \in X/G : x \in X \text{ and } g \cdot x = x \text{ for some } g \in G \setminus \{\mathrm{id}_X\}\right\}.$$

In order to investigate Bott–Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient type, we prove now Theorem 1. (See [12, Section 4.d, Théorème 2.2, Section 2.c] for the case of compact complex manifolds.)

Proof of Theorem 1. We use the same argument as in the proof of [23, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of $\tilde{X}$ can be computed using either differential forms or currents, the same holds true for the Bott–Chern and Aeppli cohomologies.

Indeed, note that, for any $p, q \in \mathbb{N}$, one has the exact sequence

$$0 \to \frac{\operatorname{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right) \cap \mathcal{D}^{p,q}\tilde{X}}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}$$
$$\to \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} \to \frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q+1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)},$$

where the maps are induced by the identity. By [1, Theorem 1], one has

$$\frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q+1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)} \simeq \frac{\ker\left(\mathrm{d}\colon \left(\wedge^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\wedge^{p+q+1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)}{\operatorname{im}\left(\mathrm{d}\colon \left(\wedge^{p+q-1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\wedge^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right)},$$

therefore it suffices to prove that the space

$$\frac{\operatorname{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right) \to \left(\mathcal{D}^{p+q}\tilde{X} \otimes_{\mathbb{R}} \mathbb{C}\right)\right) \cap \mathcal{D}^{p,q}\tilde{X}}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}$$

can be computed using just differential forms on $\tilde{X}$.

Firstly, we note that, since, by [7, p. 807],

$$\frac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X} \to \mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X} \to \mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \simeq \frac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X} \to \wedge^{\bullet,\bullet}\tilde{X}\right)},$$

one has that, if $\psi \in \wedge^{r,s}\tilde{X}$ is a $\overline{\partial}$-closed differential form, then every solution $\phi \in \mathcal{D}^{r,s-1}$ of $\overline{\partial}\phi = \psi$ is a differential form up to $\overline{\partial}$-exact terms. Indeed, since $[\psi] = 0$ in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s}\tilde{X}}{\operatorname{im}\overline{\partial}}$ and hence in $\frac{\ker\overline{\partial}\cap\wedge^{r,s}\tilde{X}}{\operatorname{im}\overline{\partial}}$, there is a differential form $\alpha \in \wedge^{r,s-1}\tilde{X}$ such that $\psi = \overline{\partial}\alpha$. Hence, $\phi - \alpha \in \mathcal{D}^{r,s-1}\tilde{X}$ defines a class in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s-1}\tilde{X}}{\operatorname{im}\overline{\partial}} \simeq \frac{\ker\overline{\partial}\cap\wedge^{r,s-1}\tilde{X}}{\operatorname{im}\overline{\partial}}$, and hence $\phi - \alpha$ is a differential form up to a $\overline{\partial}$-exact form, and so ϕ is.

By conjugation, if $\psi \in \wedge^{r,s}\tilde{X}$ is a ∂-closed differential form, then every solution $\phi \in \mathcal{D}^{r-1,s}$ of $\partial\phi = \psi$ is a differential form up to ∂-exact terms.

Now, let

$$\omega^{p,q} = \mathrm{d}\,\eta \bmod \operatorname{im}\partial\overline{\partial} \in \frac{\operatorname{im}\mathrm{d}\,\cap\mathcal{D}^{p,q}X}{\operatorname{im}\partial\overline{\partial}}.$$

Decomposing $\eta =: \sum_{p,q}\eta^{p,q}$ in pure-type components, where $\eta^{p,q} \in \mathcal{D}^{p,q}\tilde{X}$, the previous equality is equivalent to the system

$$\begin{cases} \partial\eta^{p+q-1,0} = 0 \bmod \operatorname{im}\partial\overline{\partial} \\ \overline{\partial}\eta^{p+q-\ell,\ell-1} + \partial\eta^{p+q-\ell-1,\ell} = 0 \bmod \operatorname{im}\partial\overline{\partial} \quad \text{for } \ell \in \{1,\ldots,q-1\} \\ \overline{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q} = \omega^{p,q} \bmod \operatorname{im}\partial\overline{\partial} \\ \overline{\partial}\eta^{\ell,p+q-\ell-1} + \partial\eta^{\ell-1,p+q-\ell} = 0 \bmod \operatorname{im}\partial\overline{\partial} \quad \text{for } \ell \in \{1,\ldots,p-1\} \\ \overline{\partial}\eta^{0,p+q-1} = 0 \bmod \operatorname{im}\partial\overline{\partial}. \end{cases}$$

By the above argument, we may suppose that, for $\ell \in \{0,\ldots,p-1\}$, the currents $\eta^{\ell,p+q-\ell-1}$ are differential forms: indeed, they are differential forms up to $\overline{\partial}$-exact terms, but $\overline{\partial}$-exact terms give no contribution in the system, which is modulo $\operatorname{im}\partial\overline{\partial}$. Analogously, we may suppose that, for $\ell \in \{0,\ldots,q-1\}$, the currents $\eta^{p+q-\ell-1,\ell}$ are differential forms. Then we may suppose that $\omega^{p,q} = \overline{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q}$ is a differential form. Hence (1) is proven.

Now, we prove that, fixed a G-invariant Hermitian metric on $\tilde{X}$, the Bott–Chern cohomology of $\tilde{X}$ is isomorphic to the space of $\tilde{\Delta}_{BC}$-harmonic G-invariant forms on X. Indeed, since the elements of G commute with ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$, and hence with $\tilde{\Delta}_{BC}$, the following decomposition, [12, Théorème 2.2],

$$\wedge^{\bullet,\bullet}X = \ker\tilde{\Delta}_{BC} \oplus \partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}X \oplus \left(\partial^*\wedge^{\bullet+1,\bullet}X + \overline{\partial}^*\wedge^{\bullet,\bullet+1}X\right)$$

induces a decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} = \ker\tilde{\Delta}_{BC} \oplus \partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}\tilde{X} \oplus \left(\partial^*\wedge^{\bullet+1,\bullet}\tilde{X} + \overline{\partial}^*\wedge^{\bullet,\bullet+1}\tilde{X}\right);$$

more precisely, let $\alpha \in \wedge^{\bullet,\bullet}\tilde{X}$, that is, α is a G-invariant form on X; if α has a decomposition $\alpha = h_\alpha + \partial\overline{\partial}\beta + \left(\partial^*\gamma + \overline{\partial}^*\eta\right)$ with $h_\alpha, \beta, \gamma, \eta \in \wedge^{\bullet,\bullet}X$ such that $\tilde{\Delta}_{BC}h_\alpha = 0$, then one has

$$\begin{aligned} \alpha &= \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\alpha = \left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha\right) + \partial\overline{\partial}\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\beta\right) \\ &\quad + \left(\partial^*\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\gamma\right) + \overline{\partial}^*\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\eta\right)\right), \end{aligned}$$

where $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha$, $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\beta$, $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\gamma$, $\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\eta \in \wedge^{\bullet,\bullet}\tilde{X}$ and

$$\tilde{\Delta}_{BC}\left(\frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*h_\alpha\right) = \frac{1}{\operatorname{ord}G}\sum_{g\in G}g^*\left(\tilde{\Delta}_{BC}h_\alpha\right) = 0.$$

As regards the Aeppli cohomology, one has the decomposition, [12, Section 2.c],

$$\wedge^{\bullet,\bullet}X = \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}X + \overline{\partial}\wedge^{\bullet,\bullet-1}X\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}X,$$

and hence the decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} = \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}\tilde{X} + \overline{\partial}\wedge^{\bullet,\bullet-1}\tilde{X}\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}\tilde{X},$$

from which one gets the isomorphism $H_A^{\bullet,\bullet}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_A$.

Finally, note that the Hodge-$*$-operator $*\colon \wedge^{\bullet_1,\bullet_2}\tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$ sends $\tilde{\Delta}_{BC}$-harmonic forms to $\tilde{\Delta}_A$-harmonic forms, and hence it induces an isomorphism

$$*\colon H_{BC}^{\bullet_1,\bullet_2}\left(\tilde{X}\right) \stackrel{\simeq}{\to} H_A^{n-\bullet_2,n-\bullet_1}\left(\tilde{X}\right),$$

concluding the proof. □

Remark 5. We note that another proof of the isomorphism

$$H_{BC}^{p,q}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)},$$

and a proof of the isomorphism

$$H_{A}^{p,q}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-1,q}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}$$

follow from the sheaf-theoretic interpretation of the Bott–Chern and Aeppli cohomologies, developed by J.-P. Demailly, [24, Section V I.12.1] and M. Schweitzer, [12, Section 4]; see also [25, Section 3.2].

More precisely, we recall that, for any $p, q \in \mathbb{N}$, the complex $\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right) : \mathcal{A}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots,$$

and the complex $\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right) : \mathcal{D}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots,$$

where pr denotes the projection onto the appropriate space.

Take ϕ a germ of a d-closed k-form on $\tilde{X}$, with $k \in \mathbb{N} \setminus \{0\}$, that is, a germ of a G-invariant k-form on X; by the Poincaré lemma, see, e.g., [24, I.1.22], there exists ψ a germ of a $(k-1)$-form on X such that $\phi = \mathrm{d}\,\psi$; since ϕ is G-invariant, one has

$$\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\mathrm{d}\,\psi\right) = \mathrm{d}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right),$$

that is, taking the germ of the G-invariant $(k-1)$-form

$$\tilde{\psi} := \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(k-1)$-form on $\tilde{X}$ such that $\phi = \mathrm{d}\,\tilde{\psi}$. As regards the case $k = 0$, one has straightforwardly that every (G-invariant) d-closed function on X is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents; see, e.g., [24, Theorem I.2.24].

Analogously, take ϕ a germ of a $\overline{\partial}$-closed (p, q)-form (respectively, bidimension-(p, q)-current) on $\tilde{X}$, with $q \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant (p, q)-form (respectively, bidimension-(p, q)-current) on X; by the Dolbeault and Grothendieck lemma, see, e.g., [24, I.3.29], there exists ψ a germ of a $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current) on X such that $\phi = \overline{\partial}\psi$; since ϕ is G-invariant, one has

$$\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi = \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\overline{\partial}\psi\right) = \overline{\partial}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right),$$

that is, taking the germ of the G-invariant $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current)

$$\tilde{\psi} := \frac{1}{\operatorname{ord} G} \sum_{g \in G} g^* \psi$$

on X, one gets a germ of a $(p, q-1)$-form (respectively, bidimension-$(p, q-1)$-current) on $\tilde{X}$ such that $\phi = \overline{\partial} \tilde{\psi}$. As regards the case $q = 0$, one has that every (G-invariant) $\overline{\partial}$-closed bidimension-$(p, 0)$-current on X is locally a holomorphic p-form; see, e.g., [24, I.3.29].

By the Poincaré lemma and the Dolbeault and Grothendieck lemma, one gets M. Schweitzer's lemma [12, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick.

As in [24, Lemma VI.12.1], see also [25, Proposition 3.4.1], the map

$$\left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right) \to \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{M}^{\bullet}_{\tilde{X}\,p,q}}\right)$$

of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [24, Section IV.12.6], for every $\ell \in \mathbb{N}$,

$$\mathbb{H}^{\ell}\left(\tilde{X};\ \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \mathbb{H}^{\ell}\left(\tilde{X};\ \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right).$$

Since, for every $k \in \mathbb{N}$, the sheaves $\mathcal{L}^{k}_{\tilde{X}\,p,q}$ and $\mathcal{M}^{k}_{\tilde{X}\,p,q}$ are fine (indeed, they are sheaves of $\left(\mathcal{C}^{\infty}_{\tilde{X}} \otimes_{\mathbb{R}} \mathbb{C}\right)$-modules over a paracompact space, see [6, item 5 at p. 807]), one has, see, e.g., [24, IV.4.19, (IV.12.9)],

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\ \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\colon \wedge^{p,q}\tilde{X} \to \wedge^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \wedge^{p,q}\tilde{X} \to \wedge^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\ \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)},$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\ \left(\mathcal{L}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \wedge^{p-2,q-1}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \wedge^{p-1,q-2}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\ \left(\mathcal{M}^{\bullet}_{\tilde{X}\,p,q},\ \mathrm{d}_{\mathcal{L}^{\bullet}_{\tilde{X}\,p,q}}\right)\right) \simeq \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-2,q-1}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p-1,q-2}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right)},$$

proving the stated isomorphisms.

By considering local charts, note that the same argument can be applied for general orbifolds (possibly not given by a global-quotient), as pointed out by the referee.

3. Complex orbifolds satisfying the $\partial\overline{\partial}$-lemma

We recall that a bounded double complex $\left(K^{\bullet,\bullet},\ \mathrm{d}',\ \mathrm{d}''\right)$ of vector spaces, whose associated simple complex is $(K^{\bullet},\ \mathrm{d})$ with $\mathrm{d} := \mathrm{d}' + \mathrm{d}''$, is said *to satisfy the* $\mathrm{d}'\mathrm{d}''$*-lemma*, [11], if

$$\ker \mathrm{d}' \cap \ker \mathrm{d}'' \cap \operatorname{im} \mathrm{d} = \operatorname{im} \mathrm{d}'\mathrm{d}'';$$

other equivalent conditions are provided in [11, Lemma 5.15].

An orbifold $\tilde{X}$ is said *to satisfy the* $\partial\overline{\partial}$*-lemma* if the double complex $\left(\wedge^{\bullet,\bullet}\tilde{X},\ \partial,\ \overline{\partial}\right)$ satisfies the $\partial\overline{\partial}$-lemma, that is, if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, namely, in other words, if the natural map $H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ induced by the identity is injective.

Characterizations of compact complex manifolds satisfying the $\partial\overline{\partial}$-lemma in terms of their cohomological properties have been provided by P. Deligne, Ph.A. Griffiths, J. Morgan and D.P. Sullivan in [11, Proposition 5.17, 5.21], and by the author and A. Tomassini in [26, Theorem B]. As a corollary of their characterization, P. Deligne, Ph.A. Griffiths, J. Morgan and

D.P. Sullivan proved that, given X and Y compact complex manifolds of the same dimension and $f\colon X \to Y$ a holomorphic birational map, if X satisfies the $\partial\overline{\partial}$-lemma, then also Y satisfies the $\partial\overline{\partial}$-lemma, [11, Theorem 5.22].

In this section, we extend [11, Theorem 5.22] to the case of orbifolds, by straightforwardly adapting a result by R.O. Wells, [13, Theorem 3.1], to the orbifold case.

Theorem 6 (*See [13, Theorem 3.1]*)**.** *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. Then the map $\epsilon\colon \tilde{Y} \to \tilde{X}$ induces injective maps*

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right), \qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right), \quad \text{and} \quad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right).$$

Proof. We follow closely the proof of [13, Theorem 3.1] and adapt it to the orbifold case.

`Step 1`—*Notations*. The morphism $\epsilon\colon \tilde{Y} \to \tilde{X}$ of complex orbifolds induces morphisms

$$\epsilon^*\colon \wedge^\bullet \tilde{X} \to \wedge^\bullet \tilde{Y} \quad \text{and} \quad \epsilon^*\colon \wedge^{\bullet,\bullet} \tilde{X} \to \wedge^{\bullet,\bullet} \tilde{Y}$$

of $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces, and hence, by duality,

$$\epsilon_*\colon \mathcal{D}_\bullet\tilde{Y} \to \mathcal{D}_\bullet\tilde{X} \quad \text{and} \quad \epsilon_*\colon \mathcal{D}_{\bullet,\bullet}\tilde{Y} \to \mathcal{D}_{\bullet,\bullet}\tilde{X}.$$

Moreover, recall that, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, there are natural inclusions

$$T_\cdot\colon \wedge^\bullet X \to \mathcal{D}^\bullet X :=: \mathcal{D}_{2n-\bullet}X \quad \text{and} \quad T_\cdot\colon \wedge^{\bullet,\bullet} X \to \mathcal{D}^{\bullet,\bullet} X :=: \mathcal{D}_{n-\bullet,n-\bullet}X,$$

where n is the complex dimension of X.

Both ϵ^* and ϵ_* commute with d, ∂ and $\overline{\partial}$, and hence they induce morphisms of complexes

$$\epsilon^*_{dR}\colon \left(\wedge^\bullet \tilde{X},\, \mathrm{d}\right) \to \left(\wedge^\bullet \tilde{Y},\, \mathrm{d}\right) \quad \text{and} \quad \epsilon^{dR}_*\colon \left(\mathcal{D}^\bullet\tilde{Y},\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet\tilde{X},\, \mathrm{d}\right),$$

and, for any $p \in \mathbb{N}$,

$$\epsilon^*_{\overline{\partial}}\colon \left(\wedge^{p,\bullet} \tilde{X},\, \overline{\partial}\right) \to \left(\wedge^{p,\bullet} \tilde{Y},\, \overline{\partial}\right) \quad \text{and} \quad \epsilon^{\overline{\partial}}_*\colon \left(\mathcal{D}^{p,\bullet}\tilde{Y},\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet}\tilde{X},\, \overline{\partial}\right),$$

and, for any $p, q \in \mathbb{N}$,

$$\epsilon^*_{BC}\colon \left(\wedge^{p-1,q-1} \tilde{X} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q} \tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q} \tilde{X} \oplus \wedge^{p,q+1} \tilde{X}\right) \to \left(\wedge^{p-1,q-1} \tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q} \tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q} \tilde{Y} \oplus \wedge^{p,q+1} \tilde{Y}\right)$$

and

$$\epsilon^{BC}_*\colon \left(\mathcal{D}^{p-1,q-1}\tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{Y} \oplus \wedge^{p,q+1} \tilde{Y}\right) \to \left(\mathcal{D}^{p-1,q-1}\tilde{X} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{X} \oplus \mathcal{D}^{p,q+1}\tilde{X}\right);$$

hence, they induce morphisms between the corresponding cohomologies:

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right), \qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right), \quad \text{and} \quad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right).$$

Recall that $T_\cdot$ commutes with d, ∂ and $\overline{\partial}$, and hence it induces, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, morphisms

$$T_\cdot\colon \left(\wedge^\bullet X,\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet X,\, \mathrm{d}\right),$$

and, for any $p \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p,\bullet} X,\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet} X,\, \overline{\partial}\right),$$

and, for any $p, q \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p-1,q-1} X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q} X \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q} X \oplus \wedge^{p,q+1} X\right) \to \left(\mathcal{D}^{p-1,q-1}X \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}X \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}X \oplus \wedge^{p,q+1} X\right);$$

by [1, Theorem 1], by [7, p. 807], and by Theorem 1, these maps are in fact quasi-isomorphisms.

`Step 3`—*It holds $\epsilon_*\, T_\cdot\, \epsilon^* = \mu \cdot T_\cdot$ for some $\mu \in \mathbb{N} \setminus \{0\}$*. Indeed, consider the diagrams

$$\begin{array}{ccc} \wedge^\bullet\tilde{Y} & \xrightarrow{T_\cdot} & \mathcal{D}^\bullet\tilde{Y} \\ {\scriptstyle\epsilon^*}\uparrow & & \downarrow{\scriptstyle\epsilon_*} \\ \wedge^\bullet\tilde{X} & \xrightarrow[T_\cdot]{} & \mathcal{D}^\bullet\tilde{X} \end{array}, \qquad \text{respectively} \qquad \begin{array}{ccc} \wedge^{\bullet,\bullet}\tilde{Y} & \xrightarrow{T_\cdot} & \mathcal{D}^{\bullet,\bullet}\tilde{Y} \\ {\scriptstyle\epsilon^*}\uparrow & & \downarrow{\scriptstyle\epsilon_*} \\ \wedge^{\bullet,\bullet}\tilde{X} & \xrightarrow[T_\cdot]{} & \mathcal{D}^{\bullet,\bullet}\tilde{X} \end{array}.$$

One has that there exists a proper analytic subset $S_{\tilde{Y}}$ of $\tilde{Y} \setminus \operatorname{Sing}\left(\tilde{Y}\right)$ such that $S_{\tilde{Y}}$ has measure zero in $\tilde{Y}$ and

$$\epsilon\lfloor_{\tilde{Y}\setminus\left(\operatorname{Sing}(\tilde{Y})\cup S_{\tilde{Y}}\right)} : \tilde{Y} \setminus \left(\operatorname{Sing}\left(\tilde{Y}\right) \cup S_{\tilde{Y}}\right) \to \tilde{X} \setminus \left(\operatorname{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$$

is a finitely-sheeted covering mapping of sheeting number $\mu \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{U} :=: \{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open covering of $\tilde{X} \setminus \left(\operatorname{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$, and let $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ be an associated partition of unity. For every $\varphi, \psi \in \wedge^{\bullet,\bullet}\tilde{X}$, one has that

$$\begin{aligned}\left\langle \epsilon_* T.\, \epsilon^*\varphi,\ \psi\right\rangle &= \left\langle T.\, \epsilon^*\varphi,\ \epsilon^*\psi\right\rangle = \int_{\tilde{Y}} \epsilon^*\varphi \wedge \epsilon^*\psi = \int_{\tilde{Y}} \epsilon^*\left(\varphi\wedge\psi\right) = \int_{\tilde{Y}-\left(\operatorname{Sing}(\tilde{Y})\cup S_{\tilde{Y}}\right)} \epsilon^*\left(\varphi\wedge\psi\right) \\ &= \sum_{\alpha\in\mathcal{A}} \int_{\pi^{-1}(U_\alpha)} \epsilon^*\left(\rho_\alpha\left(\varphi\wedge\psi\right)\right) = \sum_{\alpha\in\mathcal{A}} \sum_{\sharp\left\{U\in\mathcal{U}:\pi^{-1}(U)=\pi^{-1}(U_\alpha)\right\}} \int_{U_\alpha} \rho_\alpha\left(\varphi\wedge\psi\right) \\ &= \mu\cdot\int_{\tilde{X}-\left(\operatorname{Sing}(\tilde{X})\cup\epsilon(S_{\tilde{Y}})\right)} \varphi\wedge\psi = \mu\cdot\int_{\tilde{X}}\varphi\wedge\psi = \left\langle \mu\, T.\varphi,\ \psi\right\rangle,\end{aligned}$$

and hence one gets that

$$\epsilon_*\, T.\, \epsilon^* = \mu\cdot T..$$

`Step 4`—*Conclusion*. Hence one has the diagrams

$$\begin{array}{ccc} \dfrac{\ker\left(d:\wedge^\bullet\tilde{X}\to\wedge^{\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(d:\wedge^{\bullet-1}\tilde{X}\to\wedge^{\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(d:\mathcal{D}^\bullet\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(d:\mathcal{D}^{\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)} \\ \Big\uparrow{\scriptstyle \epsilon^*_{dR}} & & \Big\downarrow{\scriptstyle \epsilon_*^{dR}} \\ \dfrac{\ker\left(d:\wedge^\bullet\tilde{Y}\to\wedge^{\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(d:\wedge^{\bullet-1}\tilde{Y}\to\wedge^{\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(d:\mathcal{D}^\bullet\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(d:\mathcal{D}^{\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)} \end{array},$$

such that

$$\epsilon_*^{dR}\, T.\, \epsilon^*_{dR} = \mu\cdot T.,$$

and

$$\begin{array}{ccc} \dfrac{\ker\left(\overline{\partial}:\wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\overline{\partial}:\wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\overline{\partial}:\mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(d:\mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\ \Big\uparrow{\scriptstyle \epsilon^*_{\overline{\partial}}} & & \Big\downarrow{\scriptstyle \epsilon_*^{\overline{\partial}}} \\ \dfrac{\ker\left(\overline{\partial}:\wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(\overline{\partial}:\wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\overline{\partial}:\mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(\overline{\partial}:\mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)} \end{array},$$

such that

$$\epsilon_*^{\overline{\partial}}\, T.\, \epsilon^*_{\overline{\partial}} = \mu\cdot T.,$$

and

$$\begin{array}{ccc} \dfrac{\ker\left(\partial\overline{\partial}:\wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet+1,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\partial:\wedge^{\bullet-1,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)+\operatorname{im}\left(\overline{\partial}:\wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\partial\overline{\partial}:\mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{X}\right)}{\operatorname{im}\left(\partial:\mathcal{D}^{\bullet-1,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)+\operatorname{im}\left(d:\mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\ \Big\uparrow{\scriptstyle \epsilon^*_{BC}} & & \Big\downarrow{\scriptstyle \epsilon_*^{BC}} \\ \dfrac{\ker\left(\partial\overline{\partial}:\wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(\partial:\wedge^{\bullet-1,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)+\operatorname{im}\left(\overline{\partial}:\wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\partial\overline{\partial}:\mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\operatorname{im}\left(\partial:\mathcal{D}^{\bullet-1,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)+\operatorname{im}\left(\overline{\partial}:\mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)} \end{array},$$

such that

$$\epsilon_*^{BC}\, T.\, \epsilon^*_{BC} = \mu\cdot T..$$

Since $T.$ are isomorphisms in cohomology, one gets that

$$\epsilon^*_{dR} : H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right), \qquad \epsilon^*_{\overline{\partial}} : H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right), \quad \text{and} \quad \epsilon^*_{BC} : H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)$$

are injective. □

Now, as a corollary, we can prove Theorem 2.

Proof of Theorem 2. One has the commutative diagram

$$\begin{array}{ccc} H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) & \xrightarrow[1:1]{\epsilon^*_{BC}} & H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \\ \downarrow \mathrm{id}^*_{\tilde{X}} & & 1:1 \downarrow \mathrm{id}^*_{\tilde{Y}} \\ H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) & \xrightarrow[1:1]{\epsilon^*_{dR}} & H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right) \end{array}$$

where $\mathrm{id}^*_{\tilde{X}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ and $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are the natural maps induced in the cohomology by the identity. Since $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ is injective by the assumption that $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-lemma, and $\epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)$ and $\epsilon^*_{dR}\colon H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) \to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are injective by Theorem 6, we get that also $\mathrm{id}^*_{\tilde{X}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ is injective, and hence $\tilde{X}$ satisfies the $\partial\overline{\partial}$-lemma. □

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