Cohomologies of certain orbifolds

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ABSTRACT

We study the Bott–Chern cohomology of complex orbifolds obtained as a quotient of a compact complex manifold by a finite group of biholomorphisms.

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Introduction

In order to investigate cohomological aspects of compact complex non-Kähler manifolds, and in particular with the aim to get results allowing to construct new examples of non-Kähler manifolds, we study the cohomology of complex orbifolds.

Namely, an orbifold (or V-manifold, as introduced by I. Satake, [1]) is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, for finite subgroups $G \subset \text{GL}(n; \mathbb{C})$, where $n$ is the complex dimension; in other words, local geometry of orbifolds reduces to local $G$-invariant geometry. A special case is provided by orbifolds of global-quotient type, namely, by orbifolds $\bar{X} = X/G$, where $X$ is a complex manifold and $G$ is a finite group of biholomorphisms of $X$; such orbifolds have been studied, among others, by D.D. Joyce in constructing examples of compact manifolds with special holonomy, see [2–5]. As proven by I. Satake, and W.L. Baily, from the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools for the study of the de Rham and the Dolbeault cohomology of complex orbifolds, [1,6,7].

In particular, useful tools in studying the cohomological properties of non-Kähler manifolds are provided by the Bott–Chern cohomology, that is, the bi-graded algebra

$$H^*_{BC}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\text{im } \partial \overline{\partial}},$$

and by the Aeppli cohomology, that is, the bi-graded $H^*_{BC}(X)$-module

$$H^*_{A}(X) := \frac{\ker \partial \overline{\partial}}{\text{im } \partial + \text{im } \overline{\partial}}.$$
After R. Bott and S.S. Chern, [8], and A. Aeppli, [9], several authors studied the Bott–Chern and Aeppli cohomologies in many different contexts: for example, they have been recently considered by L.-S. Tseng and S.-T. Yau in the framework of Generalized Geometry and type Iı String Theory, [10]. While for compact Kähler manifolds X one has that the Bott–Chern cohomology and the Aeppli cohomology are naturally isomorphic to the Dolbeault cohomology, [11, Lemma 5.15, Remarks 5.16, 5.21, Lemma 5.11], in general, for compact non-Kähler manifolds X, the natural maps \( H^*_{BC}(X) \to H^*_p(X) \) and \( H^*_{BC}(X) \to H^*_q(X; \mathbb{C}) \), and \( H^*_{BC}(X) \to H^*_p(X) \) and \( H^*_{BC}(X; \mathbb{C}) \to H^*_q(X) \) induced by the identity are neither injective nor surjective. One says that a compact complex manifold satisfies the \( \partial \bar{\partial} \)-lemma if every \( \partial \bar{\partial} \)-closed \( \partial \bar{\partial} \)-exact form is \( \partial \bar{\partial} \)-exact, that is, if the natural map \( H^*_{BC}(X) \to H^*_p(X; \mathbb{C}) \) is injective; compact Kähler manifolds provide the main examples of complex manifolds satisfying the \( \partial \bar{\partial} \)-lemma, [11, Lemma 5.11], other than motivations for their study.

In this note, we study the Bott–Chern cohomology and the Aeppli cohomology of compact complex orbifolds \( \tilde{X} = X/G \) of global-quotient type (where \( X \) is a compact complex manifold and \( G \) is a finite group of biholomorphisms of \( X \)), that is,

\[
H^*_{BC}(\tilde{X}) := \frac{\ker \partial \cap \ker \overline{\partial}}{\ker \partial + \ker \overline{\partial}}, \quad \text{and} \quad H^*_A(\tilde{X}) := \frac{\ker \partial \overline{\partial}}{\ker \partial + \ker \overline{\partial}},
\]

where \( \partial : \wedge^{p,q} \tilde{X} \to \wedge^{p+1,q} \tilde{X} \) and \( \overline{\partial} : \wedge^{p,q} \tilde{X} \to \wedge^{p,q+1} \tilde{X} \), and \( \wedge \cdot \cdot \cdot \overline{\partial} \) is the bi-graded \( \mathbb{C} \)-vector space of differential forms on \( \tilde{X} \), that is, of \( G \)-invariant differential forms on \( X \). We prove the following result in Section 2.

**Theorem 1.** Let \( \tilde{X} = X/G \) be a compact complex orbifold of complex dimension \( n \), where \( X \) is a compact complex manifold and \( G \) is a finite group of biholomorphisms of \( X \). For any \( p, q \in \mathbb{N} \), there is a canonical isomorphism

\[
H^*_{BC}(\tilde{X}) \cong \frac{\ker \partial \cap \ker \overline{\partial}}{\ker \partial + \ker \overline{\partial}}, \quad \text{and} \quad H^*_A(\tilde{X}) \cong \ker \overline{\partial}.
\]

where \( \partial \) and \( \overline{\partial} \) are the 4th order self-adjoint elliptic differential operators

\[
\Delta_{BC} := (\partial \overline{\partial} + (\partial \overline{\partial})^*) (\overline{\partial} \partial + (\overline{\partial} \partial)^*) + \overline{\partial} \partial + \overline{\partial} \partial \in \text{End}(\wedge \cdot \cdot \cdot \overline{\partial})
\]

and

\[
\Delta_A := \partial \overline{\partial}^* + \overline{\partial} \partial^* + (\overline{\partial} \partial)^* (\overline{\partial} \partial)^* + (\overline{\partial} \partial)^* (\overline{\partial} \partial)^* + (\overline{\partial} \partial)^* (\overline{\partial} \partial)^* \in \text{End}(\wedge \cdot \cdot \cdot \overline{\partial}).
\]

In particular, the Hodge-\( \ast \)-operator induces an isomorphism

\[
H^*_{BC}(\tilde{X}) \cong H^*_{A-2n}(\tilde{X}).
\]

We notice that the previous theorem in the case of compact complex manifolds has been proven by M. Schweitzer in [12]. As regards the \( \partial \overline{\partial} \)-lemma for complex orbifolds, by adapting a result by R.O. Wells in [13], we get the following result.

**Theorem 2.** Let \( \tilde{Y} \) and \( \tilde{X} \) be compact complex orbifolds of the same complex dimension, and let \( \epsilon : \tilde{Y} \to \tilde{X} \) be a proper surjective morphism of complex orbifolds. If \( \tilde{Y} \) satisfies the \( \partial \overline{\partial} \)-lemma, then also \( \tilde{X} \) satisfies the \( \partial \overline{\partial} \)-lemma.

### 1. Preliminaries on orbifolds

The notion of orbifolds has been introduced by I. Satake in [1], with the name of \( V \)-manifold, and has been studied, among many others, by W.L. Baily, [6,7].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, referring to [14,5,16,7].

A complex orbifold of complex dimension \( n \) is a singular complex space whose singularities are locally isomorphic to quotient singularities \( \mathbb{C}^n/G \), for finite subgroups \( G \subset \text{GL}(n; \mathbb{C}) \), [1, Definition 2].

By definition, an object (e.g., a differential form, a Riemannian metric, a Hermitian metric) on a complex orbifold \( \tilde{X} \) is defined locally at \( x \in \tilde{X} \) as a \( G_x \)-invariant object on \( \mathbb{C}^n \), where \( G_x \subset \text{GL}(n; \mathbb{C}) \) is such that \( \tilde{X} \) is locally isomorphic to \( \mathbb{C}^n/G_x \) at \( x \).
Given $\tilde{X}$ and $\tilde{Y}$ complex orbifolds, a morphism $f: \tilde{Y} \to \tilde{X}$ of complex orbifolds is a morphism of complex spaces given, locally at $y \in \tilde{Y}$, by a map $C^n/H_y \to C^n/G_{f(y)}$, where $\tilde{Y}$ is locally isomorphic to $C^n/H_y$ at $y$ and $\tilde{X}$ is locally isomorphic to $C^n/G_{f(y)}$ at $f(y)$.

In particular, one gets a differential complex $\bigwedge^* \tilde{X}$, and a double complex $\bigwedge^*, \partial, \overline{\partial}$). Define the de Rham, Dolbeault, Bott–Chern, and Aeppli cohomology groups of $\tilde{X}$ respectively as

$$H^{\text{deR}}_{\text{dr}}(\tilde{X}; \mathbb{C}) := \ker \frac{\partial}{\text{im} \overline{\partial}}, \quad H^{\text{deR}}_{\text{dr}}(\tilde{X}) := \ker \overline{\partial} \frac{\text{im} \partial}{\text{im} \overline{\partial}}.$$

The structure of double complex of $\bigwedge^*, \partial, \overline{\partial}$ induces naturally a spectral sequence $\{ (E^{p,q}_{r}, \partial_p) \}_{r \in \mathbb{N}}$, called Hodge and Frölicher spectral sequence of $\tilde{X}$, such that $E^0_{r} \simeq H^p_{\text{dr}}(\tilde{X})$ (see, e.g., [15, Section 2.4]). Hence, one has the Frölicher inequality, see [16, Theorem 2],

$$\sum_{p+q=k} \dim \mathbb{C} \geq \dim \mathbb{C}.$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde{X}$ of complex dimension $n$, one can consider the $\mathbb{R}$-linear Hodge-$*$-operator $\ast_\mathbb{R}: \bigwedge^* \tilde{X} \to \bigwedge^{2n-*} \tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\Delta := [d^*, d^*]: d^* + d^* \in \text{End} (\bigwedge^* \tilde{X})$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde{X}$ of complex dimension $n$, one can consider the $\mathbb{C}$-linear Hodge-$*$-operator $\ast_\mathbb{C}: \bigwedge^* \tilde{X} \to \bigwedge^{2n-*} \tilde{X}$, and hence the 2nd order self-adjoint elliptic differential operator $\Delta := [\overline{\partial}^*, \overline{\partial}^*]: \overline{\partial}^* + \overline{\partial}^* \in \text{End} (\bigwedge^* \tilde{X})$.

As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension $n$, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\mathbb{R}_{\tilde{X}}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $C^\infty_{\tilde{X}}$, the sheaf of germs of $(p, q)$-forms (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^p_{\tilde{X}}$, the sheaf of germs of $k$-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^k_{\tilde{X}}$, the sheaf of germs of bidimension-$(p, q)$-currents (for $p, q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}^{p,q}_{\tilde{X}} := \mathcal{D}^{p,q}_{\tilde{X}}$, the sheaf of germs of dimension-$k$-currents (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}^k_{\tilde{X}} := \mathcal{D}^k_{\tilde{X}}$, and the sheaf of holomorphic $p$-forms (for $p \in \mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a compact complex orbifold, has been proven by I. Satake, [11], and by W.L. Baily, [6].

**Theorem 3** ([1, Theorem 1], [6, Theorem H]). Let $\tilde{X}$ be a compact complex orbifold of complex dimension $n$. There is a canonical isomorphism

$$H^*_{\text{dr}}(\tilde{X}; \mathbb{R}) \simeq H^*_{\text{dr}}(\tilde{X}; \mathbb{R}).$$

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^*_{\text{dr}}(\tilde{X}; \mathbb{R}) \simeq \ker \Delta.$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^*_{\text{dr}}(\tilde{X}; \mathbb{R}) \simeq H^{2n-*}_{\text{dr}}(\tilde{X}; \mathbb{R}).$$

The isomorphism $H^*_{\text{dr}}(\tilde{X}; \mathbb{R}) \simeq \ker \Delta$ can be seen as a consequence of a more general decomposition theorem on compact orbifolds, [6, Theorem D], which holds for 2nd order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result by W.L. Baily, [7,6], holds.
Theorem 4 ([7, p. 807], [6, Theorem K]). Let $\tilde{X}$ be a compact complex orbifold of complex dimension $n$. There is a canonical isomorphism

$$H^{*,*}_{\pi}(\tilde{X}) \simeq H^{*,*}_{\pi}(\tilde{X}; \Omega^{*,*}_{\tilde{X}}).$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{*,*}_{\pi}(\tilde{X}) \simeq \ker.$$ 

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{*,*}_{\pi}(\tilde{X}) \simeq H^{n-*,*}_{\pi}(\tilde{X}).$$

2. Bott–Chern cohomology of complex orbifolds of global-quotient type

Compact complex orbifolds of the type $\tilde{X} = X/G$, where $X$ is a compact complex manifold and $G$ is a finite group of biholomorphisms of $X$, constitute one of the simplest examples of singular manifolds: more precisely, in this section, we study the Bott–Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the $G$-invariant $\Delta_{BC}$-harmonic forms on $X$, Theorem 1.

Consider

$$\tilde{X} = X/G,$$

where $X$ is a compact complex manifold and $G$ is a finite group of biholomorphisms of $X$: by the Bochner linearization theorem, [18, Theorem 1], see also [19, Theorem 1.7.2], $\tilde{X}$ turns out to be an orbifold as in I. Satake’s definition.

Such orbifolds of global-quotient type have been considered and studied by D.D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy $G_2$, [2] and [5, Chapters 11–12], and examples of compact 8-dimensional manifolds with holonomy $Spin(7)$, [3,4] and [5, Chapters 13–14]. See also [20,21] for the use of orbifolds of global-quotient type to construct a compact 8-dimensional simply-connected non-formal symplectic manifold (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I.K. Babenko and I.A. Taimanov, [22, Problem].

Since $G$ is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

$$\operatorname{Sing}(\tilde{X}) = \{ x \in X/G : x \in X \text{ and } g \cdot x = x \text{ for some } g \in G \setminus \{ \text{id}_X \} \}.$$ 

In order to investigate Bott–Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient type, we prove now Theorem 1. (See [12, Section 4.d, Théorème 2.2, Section 2.c] for the case of compact complex manifolds.)

**Proof of Theorem 1.** We use the same argument as in the proof of [23, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of $\tilde{X}$ can be computed using either differential forms or currents, the same holds true for the Bott–Chern and Aeppli cohomologies.

Indeed, note that, for any $p, q \in \mathbb{N}$, one has the exact sequence

$$0 \to \frac{\operatorname{im} \left( d \colon (D^{p,q-1}\tilde{X} \otimes \mathbb{C}) \to (D^{p+q-1}\tilde{X} \otimes \mathbb{C}) \right) \cap D^{p,q} \tilde{X}}{\ker \left( d \colon (D^{p,q} \tilde{X} \otimes \mathbb{C}) \to (D^{p+q+1}\tilde{X} \otimes \mathbb{C}) \right)} \to \frac{\operatorname{im} \left( d \colon (D^{p,q} \tilde{X} \otimes \mathbb{C}) \to (D^{p+q+1}\tilde{X} \otimes \mathbb{C}) \right)}{\ker \left( d \colon (D^{p,q-1}\tilde{X} \otimes \mathbb{C}) \to (D^{p+q-1}\tilde{X} \otimes \mathbb{C}) \right)},$$

where the maps are induced by the identity. By [1, Theorem 1], one has

$$\frac{\ker \left( d \colon (D^{p,q} \tilde{X} \otimes \mathbb{C}) \to (D^{p+q+1}\tilde{X} \otimes \mathbb{C}) \right)}{\operatorname{im} \left( d \colon (D^{p,q-1}\tilde{X} \otimes \mathbb{C}) \to (D^{p+q-1}\tilde{X} \otimes \mathbb{C}) \right)} \simeq \frac{\ker \left( d \colon (\wedge^{p,q} \tilde{X} \otimes \mathbb{C}) \to (\wedge^{p+q+1}\tilde{X} \otimes \mathbb{C}) \right)}{\operatorname{im} \left( d \colon (\wedge^{p,q-1}\tilde{X} \otimes \mathbb{C}) \to (\wedge^{p+q-1}\tilde{X} \otimes \mathbb{C}) \right)},$$

therefore it suffices to prove that the space

$$\frac{\operatorname{im} \left( d \colon (D^{p,q-1}\tilde{X} \otimes \mathbb{C}) \to (D^{p+q-1}\tilde{X} \otimes \mathbb{C}) \right) \cap D^{p,q} \tilde{X}}{\operatorname{im} \left( d \colon (D^{p-1,q-1}\tilde{X}) \to D^{p,q} \tilde{X} \right)}$$

can be computed using just differential forms on $\tilde{X}$. 


Firstly, we note that, since, by [7, p. 807],
\[
\frac{\ker \left( \overline{\partial} : \mathcal{D}^r \ast \tilde{X} \to \mathcal{D}^{r+1} \ast \tilde{X} \right)}{\text{im} \left( \overline{\partial} : \mathcal{D}^r \ast \tilde{X} \to \mathcal{D}^r \ast \tilde{X} \right)} \simeq \frac{\ker \left( \overline{\partial} : \mathcal{D}^r \ast \tilde{X} \to \mathcal{D}^r \ast \tilde{X} \right)}{\text{im} \left( \overline{\partial} : \mathcal{D}^r \ast \tilde{X} \to \mathcal{D}^r \ast \tilde{X} \right)},
\]
one has that, if \( \psi \in \wedge^{r,s} \tilde{X} \) is a \( \overline{\partial} \)-closed differential form, then every solution \( \phi \in \mathcal{D}^{r,s} \) of \( \overline{\partial} \phi = \psi \) is a differential form up to \( \overline{\partial} \)-exact terms. Indeed, since \( \psi = 0 \) in \( \ker \overline{\partial} \mathcal{D}^{r+1} \) and hence in \( \ker \overline{\partial} \mathcal{D}^{r,s} \), there is a differential form \( \alpha \in \wedge^{r,s} \tilde{X} \) such that \( \psi = \overline{\partial} \alpha \). Hence, \( \phi - \alpha \in \mathcal{D}^{r,s} \) defines a class in \( \frac{\ker \overline{\partial} \mathcal{D}^{r,s+1} \mathcal{X}}{\ker \overline{\partial} \mathcal{D}^{r,s} \mathcal{X}} \simeq \frac{\ker \overline{\partial} \mathcal{D}^{r,s+1} \tilde{X}}{\ker \overline{\partial} \mathcal{D}^{r,s} \tilde{X}} \), and hence \( \phi - \alpha \) is a differential form up to a \( \overline{\partial} \)-exact form, and so \( \phi \) is.

By conjugation, if \( \psi \in \wedge^{r,s} \tilde{X} \) is a \( \overline{\partial} \)-closed differential form, then every solution \( \phi \in \mathcal{D}^{r-s} \) of \( \overline{\partial} \phi = \psi \) is a differential form up to \( \overline{\partial} \)-exact terms.

Now, let
\[
\omega^{p,q} = \im \eta \mod \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X}.
\]
Decomposing \( \eta =: \sum_{p,q} \eta^{p,q} \) in pure-type components, where \( \eta^{p,q} \in \mathcal{D}^{p,q} \tilde{X} \), the previous equality is equivalent to the system
\[
\begin{align*}
\partial \eta^{p+q+1,0} &= 0 \mod \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} \\
\overline{\partial} \eta^{p+q-\ell,\ell-1} + \partial \eta^{p+q-\ell,\ell-1} &= 0 \mod \overline{\partial} \mathcal{D}^{p+q-\ell,\ell-1} \mathcal{X} \quad \text{for } \ell \in \{1, \ldots, q-1\} \\
\overline{\partial} \eta^{p+q-\ell-1,\ell} + \partial \eta^{p+q-\ell-1,\ell} &= \omega^{p,q} \mod \overline{\partial} \mathcal{D}^{p+q-\ell-1,\ell} \mathcal{X} \quad \text{for } \ell \in \{1, \ldots, p-1\} \\
\overline{\partial} \eta^{p+q+1,0} &= 0 \mod \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X},
\end{align*}
\]
By the above argument, we may suppose that, for \( \ell \in \{0, \ldots, p-1\} \), the currents \( \eta^{p+q-\ell,\ell-1} \) are differential forms: indeed, they are differential forms up to \( \overline{\partial} \)-exact terms, but \( \overline{\partial} \)-exact terms give no contribution in the system, which is modulo \( \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} \). Analogously, we may suppose that, for \( \ell \in \{0, \ldots, q-1\} \), the currents \( \eta^{p+q-\ell,\ell} \) are differential forms. Then we may suppose that \( \omega^{p,q} = \overline{\partial} \eta^{p+q-\ell,\ell} \) is a differential form. Hence (1) is proven.

Now, we prove that, fixed a \( G \)-invariant Hermitian metric on \( \tilde{X} \), the Bott–Chern cohomology of \( \tilde{X} \) is isomorphic to the space of \( \Delta_{BC} \)-harmonic \( G \)-invariant forms on \( X \). Indeed, since the elements of \( G \) commute with \( \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} \), and \( \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} / \overline{\partial} \mathcal{D}^{p,q} \mathcal{X} \), and hence with \( \Delta_{BC} \), the following decomposition, [12, Théorème 2.2],
\[
\wedge^{\ast \ast} \mathcal{X} = \ker \Delta_{BC} \oplus \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \oplus \left( \overline{\partial}^{\ast} \wedge^{\ast \ast} \mathcal{X} + \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \right)
\]
induces a decomposition
\[
\wedge^{\ast \ast} \tilde{X} = \ker \Delta_{BC} \oplus \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \oplus \left( \overline{\partial}^{\ast} \wedge^{\ast \ast} \tilde{X} + \overline{\partial} \mathcal{D}^{\ast \ast} \tilde{X} \right);
\]
more precisely, let \( \alpha \in \wedge^{\ast \ast} \tilde{X} \), that is, \( \alpha \) is a \( G \)-invariant form on \( X \); if \( \alpha \) has a decomposition \( \alpha = h_{\alpha} + \overline{\partial} \beta + (\overline{\partial}^{\ast} \gamma + \overline{\partial} \eta) \)
with \( h_{\alpha}, \beta, \gamma, \eta \in \wedge^{\ast \ast} \tilde{X} \) such that \( \Delta_{BC} h_{\alpha} = 0 \), then one has
\[
\alpha = \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \alpha = \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} h_{\alpha} + \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \left( \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \beta \right)
+ \left( \overline{\partial}^{\ast} \left( \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \gamma \right) + \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \left( \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \eta \right) \right),
\]
where \( \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} h_{\alpha}, \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \beta, \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \gamma, \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \eta \in \wedge^{\ast \ast} \tilde{X} \) and
\[
\Delta_{BC} \mathcal{D}^{\ast \ast} \mathcal{X} \left( \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} h_{\alpha} \right) = \frac{1}{\text{ord } G} \sum_{g \in G} g^{\ast} \left( \Delta_{BC} h_{\alpha} \right) = 0.
\]
As regards the Aeppli cohomology, one has the decomposition, [12, Section 2.2],
\[
\wedge^{\ast \ast} \mathcal{X} = \ker \Delta_{A} \oplus \left( \overline{\partial} \wedge^{\ast \ast} \mathcal{X} + \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \right) \oplus \left( \overline{\partial} \mathcal{D}^{\ast \ast} \mathcal{X} \right) \wedge^{\ast \ast} \mathcal{X}.
\]
and hence the decomposition
\[ \wedge^\bullet \bar{\Delta}_A \oplus \left( \partial \wedge^\bullet - 1 \bar{\Delta}_A + \bar{\partial} \wedge^\bullet - 1 \bar{\Delta}_A \right) \oplus \left( \partial \bar{\partial} \right)^* \wedge^{\bullet + 1} \bar{\Delta}_A, \]
from which one gets the isomorphism \( H_A^{\bullet,\bullet} \left( \bar{\Delta} \right) \simeq \ker \Delta_A \).

Finally, note that the Hodge-\( \ast \)-operator \( \ast : \wedge^\bullet \bar{\Delta}_A \rightarrow \wedge^{\bullet - 2, n-\bullet} \bar{\Delta}_A \) sends \( \bar{\Delta}_A \)-harmonic forms to \( \bar{\Delta}_A \)-harmonic forms, and hence it induces an isomorphism
\[ \ast : H_A^{\bullet,\bullet} \left( \bar{\Delta} \right) \simeq H_A^{\bullet - 2, n-\bullet} \left( \bar{\Delta} \right), \]
concluding the proof. \( \square \)

**Remark 5.** We note that another proof of the isomorphism
\[ H_{BC}^{p,q} \left( \bar{\Delta} \right) \simeq \frac{\ker \left( \partial : \mathcal{D}^{p,q} \bar{\Delta} \rightarrow \mathcal{D}^{p+1,q} \bar{\Delta} \right) \cap \ker \left( \bar{\partial} : \mathcal{D}^{p,q} \bar{\Delta} \rightarrow \mathcal{D}^{p,q+1} \bar{\Delta} \right)}{\im \left( \partial \bar{\partial} : \mathcal{D}^{p,q} \bar{\Delta} \rightarrow \mathcal{D}^{p,q} \bar{\Delta} \right)}, \]
and a proof of the isomorphism
\[ H_A^{p,q} \left( \bar{\Delta} \right) \simeq \frac{\ker \left( \partial \bar{\partial} : \mathcal{D}^{p,q} \bar{\Delta} \rightarrow \mathcal{D}^{p+1,q+1} \bar{\Delta} \right)}{\im \left( \partial : \mathcal{D}^{p,q-1} \bar{\Delta} \rightarrow \mathcal{D}^{p,q} \bar{\Delta} \right) + \im \left( \bar{\partial} : \mathcal{D}^{p,q-1} \bar{\Delta} \rightarrow \mathcal{D}^{p,q} \bar{\Delta} \right)}, \]
follow from the sheaf-theoretic interpretation of the Bott–Chern and Aeppli cohomologies, developed by J.-P. Demailly, [24, Section V.1.12.1] and M. Schweitzer, [12, Section 4]; see also [25, Section 3.2].

More precisely, we recall that, for any \( p, q \in \mathbb{N} \), the complex \( \left( \mathcal{L}_{X,p,q}^\bullet, d_{L_{X,p,q}}^\bullet \right) \) of sheaves is defined as
\[ \left( \mathcal{M}_{X,p,q}^\bullet, d_{M_{X,p,q}}^\bullet \right) : \mathcal{A}_{X,p,q}^{0,0} \xrightarrow{\text{pr} \circ d} \mathcal{A}_{X,p,q}^{r+s} \rightarrow \cdots \]
and the complex \( \left( \mathcal{M}_{X,p,q}^\bullet, d_{M_{X,p,q}}^\bullet \right) \) of sheaves is defined as
\[ \left( \mathcal{M}_{X,p,q}^\bullet, d_{M_{X,p,q}}^\bullet \right) : \mathcal{D}_{X,p,q}^{0,0} \xrightarrow{\text{pr} \circ d} \mathcal{D}_{X,p,q}^{r+s} \rightarrow \cdots \]
where \( \text{pr} \) denotes the projection onto the appropriate space.

Take \( \phi \) a germ of a \( d \)-closed \( k \)-form on \( X \), with \( k \in \mathbb{N} \setminus \{0\} \), that is, a germ of a \( G \)-invariant \( k \)-form on \( X \); by the Poincaré lemma, see, e.g., [24, I.1.22], there exists \( \psi \) a germ of a \( (k-1) \)-form on \( X \) such that \( \phi = d \psi \); since \( \phi \) is \( G \)-invariant, one has
\[ \phi = \frac{1}{\text{ord} G} \sum_{g \in G} g^* \phi = \frac{1}{\text{ord} G} \sum_{g \in G} g^* \left( d \psi \right) = d \left( \frac{1}{\text{ord} G} \sum_{g \in G} g^* \psi \right), \]
that is, taking the germ of the \( G \)-invariant \( (k-1) \)-form
\[ \psi := \frac{1}{\text{ord} G} \sum_{g \in G} g^* \psi \]
on \( X \), one gets a germ of a \( (k-1) \)-form on \( X \) such that \( \phi = d \psi \). As regards the case \( k = 0 \), one has straightforwardly that every \( (G \)-invariant) \( d \)-closed function on \( X \) is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents; see, e.g., [24, Theorem I.2.24].

Analogously, take \( \phi \) a germ of a \( \bar{\partial} \)-closed \( (p, q) \)-form (respectively, bidimension-\( (p, q) \)-current) on \( \bar{\Delta} \), with \( q \in \mathbb{N} \setminus \{0\} \), that is, a germ of a \( G \)-invariant \( (p, q) \)-form (respectively, bidimension-\( (p, q) \)-current) on \( X \); by the Dolbeault and Grothendieck lemma, see, e.g., [24, I.3.29], there exists \( \psi \) a germ of a \( (p, q - 1) \)-form (respectively, bidimension-\( (p, q - 1) \)-current) on \( X \) such that \( \phi = \bar{\partial} \psi \); since \( \phi \) is \( G \)-invariant, one has
\[ \phi = \frac{1}{\text{ord} G} \sum_{g \in G} g^* \phi = \frac{1}{\text{ord} G} \sum_{g \in G} g^* \left( \bar{\partial} \psi \right) = \bar{\partial} \left( \frac{1}{\text{ord} G} \sum_{g \in G} g^* \psi \right), \]
that is, taking the germ of the $G$-invariant $(p, q - 1)$-form (respectively, bidimension-$(p, q - 1)$-current)

$$\tilde{\psi} := \frac{1}{\text{ord } G} \sum_{g \in G} g^* \psi$$

on $X$, one gets a germ of a $(p, q - 1)$-form (respectively, bidimension-$(p, q - 1)$-current) on $\tilde{X}$ such that $\psi = \tilde{\phi} \tilde{\psi}$. As regards the case $q = 0$, one has that every $(G$-invariant) $\tilde{\partial}$-closed bidimension-$(p, 0)$-current on $X$ is locally a holomorphic $p$-form; see, e.g., [24, 1.3.29].

By the Poincaré lemma and the Dolbeault and Grothendieck lemma, one gets M. Schweitzer’s lemma [12, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick.

As in [24, Lemma VI.12.1], see also [25, Proposition 3.4.1], the map

$$(L_{\tilde{X}, \partial}^\bullet \cdot d_{\tilde{X}, \partial}) \to (\mathcal{M}_{\tilde{X}, \partial}^\bullet \cdot d_{\mathcal{M}_{\tilde{X}, \partial}})$$

of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [24, Section IV.12.6], for every $\ell \in \mathbb{N}$,

$$\mathbb{H}^\ell (\tilde{X}; (L_{\tilde{X}, \partial}^\bullet \cdot d_{\tilde{X}, \partial})) \cong \mathbb{H}^\ell (\tilde{X}; (\mathcal{M}_{\tilde{X}, \partial}^\bullet \cdot d_{\mathcal{M}_{\tilde{X}, \partial}})).$$

Since, for every $k \in \mathbb{N}$, the sheaves $L_{\tilde{X}, \partial}^k$ and $\mathcal{M}_{\tilde{X}, \partial}^k$ are fine (indeed, they are sheaves of $(\mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C})$-modules over a paracompact space, see [6, item 5 at p. 807]), one has, see, e.g., [24, IV.4.19, (IV.12.9)],

$$\mathbb{H}^{p+q-1} (\tilde{X}; (L_{\tilde{X}, \partial}^\bullet \cdot d_{\tilde{X}, \partial})) \cong \ker \left( \tilde{\partial}; \wedge^{p+1,q} \tilde{X} \to \wedge^{p+1,q+1} \tilde{X} \right) \cong \ker \left( \tilde{\partial}; \wedge^{p,q+1} \tilde{X} \to \wedge^{p,q+1} \tilde{X} \right)$$

and

$$\mathbb{H}^{p+q-1} (\tilde{X}; (\mathcal{M}_{\tilde{X}, \partial}^\bullet \cdot d_{\mathcal{M}_{\tilde{X}, \partial}})) \cong \ker \left( \tilde{\partial}; \wedge^{p,q} \tilde{X} \to \wedge^{p,q+1} \tilde{X} \right) \cong \ker \left( \tilde{\partial}; \wedge^{p,q} \tilde{X} \to \wedge^{p,q+1} \tilde{X} \right),$$

and

$$\mathbb{H}^{p+q-2} (\tilde{X}; (L_{\tilde{X}, \partial}^\bullet \cdot d_{\tilde{X}, \partial})) \cong \ker \left( \tilde{\partial}; \wedge^{p-1,q-1} \tilde{X} \to \wedge^{p,q} \tilde{X} \right) \cong \ker \left( \tilde{\partial}; \wedge^{p-1,q-1} \tilde{X} \to \wedge^{p,q} \tilde{X} \right)$$

and

$$\mathbb{H}^{p+q-2} (\tilde{X}; (\mathcal{M}_{\tilde{X}, \partial}^\bullet \cdot d_{\mathcal{M}_{\tilde{X}, \partial}})) \cong \ker \left( \tilde{\partial}; \wedge^{p-1,q-1} \tilde{X} \to \wedge^{p,q} \tilde{X} \right) \cong \ker \left( \tilde{\partial}; \wedge^{p-1,q-1} \tilde{X} \to \wedge^{p,q} \tilde{X} \right),$$

proving the stated isomorphisms.

By considering local charts, note that the same argument can be applied for general orbifolds (possibly not given by a global-quotient), as pointed out by the referee.

### 3. Complex orbifolds satisfying the $\tilde{\partial} \tilde{\partial}$-lemma

We recall that a bounded double complex $(K^{\bullet, \bullet}, d', d'')$ of vector spaces, whose associated simple complex is $(K^{\bullet, \bullet}, d)$ with $d := d' + d''$, is said to satisfy the $d'd''$-lemma, [11], if

$$\ker d' \cap \ker d'' \cap \text{im } d = \text{im } d' d'';$$

other equivalent conditions are provided in [11, Lemma 5.15].

An orbifold $\tilde{X}$ is said to satisfy the $\tilde{\partial} \tilde{\partial}$-lemma if the double complex $(\wedge^{*,*} \tilde{X}, \partial, \tilde{\partial})$ satisfies the $\tilde{\partial} \tilde{\partial}$-lemma, that is, if every $\partial$-closed $\tilde{\partial}$-closed $d$-exact form is $\tilde{\partial} \tilde{\partial}$-exact, namely, in other words, if the natural map $H^*_BC (\tilde{X}) \to H^*DR (\tilde{X}; \mathbb{C})$ induced by the identity is injective.

Characterizations of compact complex manifolds satisfying the $\tilde{\partial} \tilde{\partial}$-lemma in terms of their cohomological properties have been provided by P. Deligne, Ph.A. Griffiths, J. Morgan and D.P. Sullivan in [11, Proposition 5.17, 5.21], and by the author and A. Tomassini in [26, Theorem B]. As a corollary of their characterization, P. Deligne, Ph.A. Griffiths, J. Morgan and...
D.P. Sullivan proved that, given $X$ and $Y$ compact complex manifolds of the same dimension and $f: X \to Y$ a holomorphic birational map, if $X$ satisfies the $\partial \bar{\partial}$-lemma, then also $Y$ satisfies the $\partial \bar{\partial}$-lemma, [11, Theorem 5.22].

In this section, we extend [11, Theorem 5.22] to the case of orbifolds, by straightforwardly adapting a result by R.O. Wells, [13, Theorem 3.1], to the orbifold case.

**Theorem 6** (See [13, Theorem 3.1]). Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon: \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. Then the map $\epsilon: \tilde{Y} \to \tilde{X}$ induces injective maps

$$
epsilon^*: \Lambda^* \tilde{X} \to \Lambda^* \tilde{Y} \quad \text{and} \quad \epsilon^*: \Lambda^{**} \tilde{X} \to \Lambda^{**} \tilde{Y}$$

of $\mathbb{R}$-vector spaces and $C$-vector spaces, and hence, by duality,

$$\epsilon_*: \mathcal{D}_* \tilde{Y} \to \mathcal{D}_* \tilde{X} \quad \text{and} \quad \epsilon_*: \mathcal{D}_{**} \tilde{Y} \to \mathcal{D}_{**} \tilde{X}.$$ Moreover, recall that, for $X \in \{\tilde{X}, \tilde{Y}\}$, there are natural inclusions

$$T: \Lambda^* X \to \mathcal{D}^* X := \mathcal{D}_{2n} X \quad \text{and} \quad T: \Lambda^{**} X \to \mathcal{D}^{**} X := \mathcal{D}_{n^2+n} X,$$

where $n$ is the complex dimension of $X$.

Both $\epsilon^*$ and $\epsilon_*$ commute with $d$, $\partial$ and $\bar{\partial}$, and hence they induce morphisms of complexes

$$\epsilon^{*}_{dr}: \Lambda^* X \to \Lambda^* \tilde{X}, \quad \partial \tilde{Y} \to \Lambda^{**} X \to \Lambda^{**} \tilde{X} \to \Lambda^{**} \tilde{Y},$$

and, for any $p \in \mathbb{N}$,

$$\epsilon^{*}_{pr}: \Lambda^p \tilde{X} \to \Lambda^p \tilde{Y} \quad \text{and} \quad \epsilon^{*}_{pr}: \mathcal{D}^p \tilde{X} \to \mathcal{D}^p \tilde{Y},$$

and, for any $p, q \in \mathbb{N}$,

$$\epsilon^{*}_{bc}: \Lambda^p \tilde{X} \to \Lambda^p \tilde{Y} \quad \text{and} \quad \epsilon^{*}_{bc}: \mathcal{D}^p \tilde{X} \to \mathcal{D}^p \tilde{Y}.$$  

hence, they induce morphisms between the corresponding cohomologies:

$$\epsilon^{*}_{dr}: H^*_{dr} (\tilde{X}; \mathbb{R}) \to H^*_{dr} (\tilde{Y}; \mathbb{R}), \quad \epsilon^{*}_{pr}: H^{**}_{pr} (\tilde{X}) \to H^{**}_{pr} (\tilde{Y}), \quad \text{and} \quad \epsilon^{*}_{bc}: H^{**}_{bc} (\tilde{X}) \to H^{**}_{bc} (\tilde{Y}).$$

Recall that $T$ commutes with $d$, $\partial$ and $\bar{\partial}$, and hence it induces, for $X \in \{\tilde{X}, \tilde{Y}\}$, morphisms

$$T: (\Lambda^* X, d) \to (\mathcal{D}^* X, d),$$

and, for any $p \in \mathbb{N}$,

$$T: (\Lambda^p \tilde{X}, \partial) \to (\mathcal{D}^p \tilde{X}, \partial),$$

and, for any $p, q \in \mathbb{N}$,

$$T: (\Lambda^p \tilde{X}, \partial) \to (\mathcal{D}^p \tilde{X}, \partial),$$

by [1, Theorem 1], by [7, p. 807], and by Theorem 1, these maps are in fact quasi-isomorphisms.

**Step 3**—It holds $\epsilon_* T. \epsilon^* = \mu \cdot T$ for some $\mu \in \mathbb{N} \setminus \{0\}$. Indeed, consider the diagrams

$$\Lambda^* \tilde{Y} \xrightarrow{T} \mathcal{D}^* \tilde{Y}, \quad \Lambda^{**} \tilde{Y} \xrightarrow{T} \mathcal{D}^{**} \tilde{Y},$$

respectively

$$\Lambda^* \tilde{X} \xrightarrow{T} \mathcal{D}^* \tilde{X}, \quad \Lambda^{**} \tilde{X} \xrightarrow{T} \mathcal{D}^{**} \tilde{X}.$$
One has that there exists a proper analytic subset $S_\gamma$ of $\tilde{Y} \setminus \text{Sing}(\tilde{Y})$ such that $S_\gamma$ has measure zero in $\tilde{Y}$ and

$$\epsilon \mid_{\text{Sing}(\tilde{Y}) \cup S_\gamma} : \tilde{Y} \setminus (\text{Sing}(\tilde{Y}) \cup S_\gamma) \to \tilde{X} \setminus \text{Sing}(\tilde{X} \cup S_\gamma)$$

is a finitely-sheeted covering mapping of sheeting number $\mu \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{U} := \{U_a \mid a \in A\}$ be an open covering of $\tilde{X} \setminus \text{Sing}(\tilde{X} \cup S_\gamma)$, and let $\{\rho_a\}_{a \in A}$ be an associated partition of unity. For every $\varphi, \psi \in \Lambda^* \tilde{X}$, one has that

$$\{\epsilon_a \cdot T \cdot \epsilon^* \varphi, \psi\} = \{T \cdot \epsilon^* \varphi, \epsilon^* \psi\} = \int_{\tilde{Y}} \epsilon^* \varphi \wedge \epsilon^* \psi = \int_{\tilde{Y}} \epsilon^* (\varphi \wedge \psi) = \int_{\tilde{Y} \setminus (\text{Sing}(\tilde{Y}) \cup S_\gamma)} \epsilon^* (\varphi \wedge \psi)$$

$$= \sum_{\alpha \in A} \int_{\pi^{-1}(U_a)} \epsilon^* (\rho_a (\varphi \wedge \psi)) = \sum_{\alpha \in A} \sum_{\{U \cap \pi^{-1}(U_a) = \pi^{-1}(U_a)\}} \int_{U_a} \rho_a (\varphi \wedge \psi)$$

$$= \mu \cdot \int_{\tilde{X} \setminus (\text{Sing}(\tilde{X}) \cup S_\gamma)} \varphi \wedge \psi = \mu \cdot \int_{\tilde{X}} \varphi \wedge \psi = \langle \mu \cdot T \varphi, \psi \rangle,$$
Now, as a corollary, we can prove Theorem 2.

**Proof of Theorem 2.** One has the commutative diagram

\[
\begin{array}{ccc}
H^{•^*}_{BC}(\tilde{X}) & \xrightarrow{\epsilon^{•^*}_{BC}} & H^{•^*}_{BC}(\tilde{Y}) \\
\downarrow \text{id}^*_X & & \downarrow \text{id}^*_Y \\
H^{•^*}_{dr}(\tilde{X};\mathbb{C}) & \xrightarrow{\epsilon^{•^*}_{dr}} & H^{•^*}_{dr}(\tilde{Y};\mathbb{C})
\end{array}
\]

where \(\text{id}^*_X: H^{•^*}_{BC}(\tilde{X}) \rightarrow H^{•^*}_{dr}(\tilde{X};\mathbb{C})\) and \(\text{id}^*_Y: H^{•^*}_{BC}(\tilde{Y}) \rightarrow H^{•^*}_{dr}(\tilde{Y};\mathbb{C})\) are the natural maps induced in the cohomology by the identity. Since \(\text{id}^*_Y: H^{•^*}_{BC}(\tilde{Y}) \rightarrow H^{•^*}_{dr}(\tilde{Y};\mathbb{C})\) is injective by the assumption that \(\tilde{Y}\) satisfies the \(\partial\bar{\partial}\)-lemma, and \(\epsilon^{•^*}_{BC}: H^{•^*}_{BC}(\tilde{X}) \rightarrow H^{•^*}_{BC}(\tilde{Y})\) and \(\epsilon^{•^*}_{dr}: H^{•^*}_{dr}(\tilde{X};\mathbb{C}) \rightarrow H^{•^*}_{dr}(\tilde{Y};\mathbb{C})\) are injective by Theorem 6, we get that also \(\text{id}^*_X: H^{•^*}_{BC}(\tilde{X}) \rightarrow H^{•^*}_{dr}(\tilde{X};\mathbb{C})\) is injective, and hence \(\tilde{X}\) satisfies the \(\partial\bar{\partial}\)-lemma. \(\square\)

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