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BOTT-CHERN COHOMOLOGY OF SOLVMANIFOLDS

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Abstract. We study conditions under which sub-complexes of a double complex of vector spaces allow to compute the Bott-Chern cohomology. We are especially aimed at studying the Bott-Chern cohomology of a special class of solvmanifolds.

Introduction

Given a double complex $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of vector spaces, both the cohomology of the associated total complex $\left(\bigoplus_{p+q=\bullet} A^{p,q},\, \partial+\overline{\partial}\right)$ and the cohomologies of the rows $(A^{\bullet,q},\, \partial)$ and of the columns $\left(A^{p,\bullet},\, \overline{\partial}\right)$ have been widely studied. Two other interesting cohomologies are the *Bott-Chern cohomology*, namely, the cohomology of the complex

$$A^{p-1,q-1} \xrightarrow{\partial\overline{\partial}} A^{p,q} \xrightarrow{\partial+\overline{\partial}} A^{p+1,q} \oplus A^{p,q+1} ,$$

and the *Aeppli cohomology*, namely, the cohomology of the complex

$$A^{p-1,q} \oplus A^{p,q-1} \xrightarrow{\left(\partial,\, \overline{\partial}\right)} A^{p,q} \xrightarrow{\partial\overline{\partial}} A^{p+q,q+1} .$$

For a compact complex manifold X, the Bott-Chern and the Aeppli cohomologies of the double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ have been studied by many authors in several contexts, see, e.g., [1, 19, 16, 29, 69, 2, 64, 47, 17, 68, 4, 10]. They appear to be a completing useful tool besides the de Rham and the Dolbeault cohomologies. In this spirit, in [10], it is shown that an inequality *à la* Frölicher, involving just the dimensions of the Bott-Chern cohomology and of the de Rham cohomology, holds true on any compact complex manifold, and further allows to characterize the validity of the $\partial\overline{\partial}$-Lemma (namely, the very special cohomological property that every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact too, see, e.g., [29]).

A compact manifold satisfies the $\partial\overline{\partial}$-Lemma if and only if the Bott-Chern cohomology is naturally isomorphic to the Dolbeault cohomology, [29, Remark 5.16]. Therefore, since compact Kähler manifolds satisfy the $\partial\overline{\partial}$-Lemma because of the Kähler identities, [29, Lemma 5.11], the Bott-Chern cohomology is particularly interesting in studying complex non-Kähler manifolds.

In non-Kähler geometry, a very fruitful source of counter-examples is provided by the class of nilmanifolds and solvmanifolds, namely, compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups. For instance, the geometry of nilmanifolds can be often reduced to the study of the associated Lie algebras, [21, 60, 14]. On the other hand, nilmanifolds do not admit too strong geometric structures, [15, 35]. More precisely, on a nilmanifold, the finite-dimensional sub-complex of left-invariant forms (namely, the forms being invariant for the action of the Lie group on itself given by left-translations) suffices in computing the de Rham cohomology, [55, 37]. Whenever the nilmanifold is endowed with a suitable left-invariant complex structure, also the Dolbeault cohomology, [61, 25, 22, 59, 60], and the Bott-Chern cohomology, [4], can be computed by means of just left-invariant forms.

Instead, for solvmanifolds, the left-invariant forms are usually not enough to recover the whole de Rham cohomology: an example is the non-completely-solvable solvmanifold provided in [27, Corollary 4.2]. The de Rham cohomology of solvmanifolds has been studied by several authors, e.g., A. Hattori [37], G. D. Mostow [53], S. Console and A. Fino [23], and the second author [39, 43]. Several results concerning the Dolbeault cohomology have been proven by the second author, [40, 43]; such results

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allow to study Hodge symmetry, Hodge decomposition, formality, and the Hodge and Frölicher spectral sequence on solvmanifolds, [41, 42, 44].

In this note, we study the Bott-Chern cohomology of a certain class of solvmanifolds. This is done with the scope to further investigate the complex geometry of non-Kähler manifolds and especially its cohomological aspects. More precisely, we start by studying conditions under which the Bott-Chern cohomology of a double complex can be completely recovered by a suitable sub-complex; see Theorem 1.3 and Theorem 1.6. As an application, we get the following result. (For further applications to the study of the symplectic cohomologies studied by L.-S. Tseng and S.-T. Yau in [66, 67], see [8].)

Theorem (see Theorem 2.16 and Theorem 2.25). *Let G be a connected simply-connected solvable Lie group admitting a co-compact discrete subgroup Γ and endowed with a G-left-invariant complex structure. If*

- *either G is a semidirect product $\mathbb{C}^n \ltimes_\phi N$ of $\mathbb{C}^n$ and a connected simply-connected nilpotent Lie group N endowed with an N-left-invariant complex structure satisfying some conditions (see Assumption 2.11),*
- *or G is a complex Lie group,*

then there is an explicit finite-dimensional sub-complex $C^{\bullet,\bullet}$ of the double complex $\left(\wedge^{\bullet,\bullet}\Gamma\backslash G\,,\, \partial,\, \overline{\partial}\right)$ which computes the Bott-Chern cohomology of the solvmanifold $\Gamma\backslash G$.

As an application, we explicitly compute the Bott-Chern cohomology of the completely-solvable Nakamura manifold and of the complex parallelizable Nakamura manifold. This gives us, as a corollary, the following result.

Theorem (see Theorem 2.20). *Satisfying the $\partial\overline{\partial}$-Lemma is not a strongly-closed property under small deformations of the complex structure.*

In fact, in [7], we prove that satisfying the $\partial\overline{\partial}$-Lemma is not a (Zariski-)closed property.

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1. Computing the cohomologies of double complexes by means of sub-complexes

In this section, we study several cohomologies associated to a bounded double complex of $\mathbb{C}$-vector spaces; in particular, we are interested in studying when such cohomologies can be recovered by means of a suitable (possibly finite-dimensional) sub-complex.

1.1. The cohomology of the associated total complex.

Let $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, namely, $\partial \in \mathrm{End}^{1,0}\left(A^{\bullet,\bullet}\right)$ and $\overline{\partial} \in \mathrm{End}^{0,1}\left(A^{\bullet,\bullet}\right)$ are such that $\partial^2 = \overline{\partial}^2 = \left[\partial, \overline{\partial}\right] = 0$, and $A^{p,q} = \{0\}$ but for finitely-many $(p,q) \in \mathbb{Z}^2$. Denote by

$$\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right) := \bigoplus_{p+q=\bullet} A^{p,q},\ \mathrm{d} := \partial + \overline{\partial}\right)$$

the total complex associated to $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$. The bi-grading of $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ induces two natural bounded filtrations of $\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\, \mathrm{d}\right)$, namely,

$$\left\{\left({}'F^p\,\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right) := \bigoplus_{\substack{r+s=\bullet\\ r\geq p}} A^{r,s},\ \mathrm{d}\lfloor_{{}'F^p\,\mathrm{Tot}^\bullet(A^{\bullet,\bullet})}\right) \hookrightarrow \left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\ \mathrm{d}\right)\right\}_{p\in\mathbb{Z}}$$

and

$$\left\{\left({}''F^q\,\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right) := \bigoplus_{\substack{r+s=\bullet\\ s\geq q}} A^{r,s},\ \mathrm{d}\lfloor_{{}''F^q\,\mathrm{Tot}^\bullet(A^{\bullet,\bullet})}\right) \hookrightarrow \left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\ \mathrm{d}\right)\right\}_{q\in\mathbb{Z}} .$$

Such filtrations induce naturally two spectral sequences, respectively,

$$\left\{\left({}'E_r^{\bullet,\bullet}\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right),\, {}'\mathrm{d}_r\right)\right\}_{r\in\mathbb{Z}} \qquad \text{and} \qquad \left\{\left({}''E_r^{\bullet,\bullet}\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right),\, {}''\mathrm{d}_r\right)\right\}_{r\in\mathbb{Z}} ,$$

such that

$$ {}'E_1^{\bullet_1,\bullet_2}\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \;\simeq\; H^{\bullet_2}\left(A^{\bullet_1,\bullet},\,\overline{\partial}\right) \;\Rightarrow\; H^{\bullet_1+\bullet_2}\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) \;, $$

and

$$ {}''E_1^{\bullet_1,\bullet_2}\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \;\simeq\; H^{\bullet_1}\left(A^{\bullet,\bullet_2},\,\partial\right) \;\Rightarrow\; H^{\bullet_1+\bullet_2}\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) \;, $$

see, e.g., [51, §2.4], see also [34, §3.5], [24, Theorem 1, Theorem 3].

One gets straightforwardly the following result, providing a sufficient condition under which a sub-complex $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ allows to recover the cohomology of $\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right)$.

Proposition 1.1. *Let $\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, and let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a sub-complex. If, for every $p \in \mathbb{Z}$, the induced map $\left(C^{p,\bullet},\,\overline{\partial}\right) \hookrightarrow \left(A^{p,\bullet},\,\overline{\partial}\right)$ of complexes is a quasi-isomorphism, then the induced map*

$$ \left(\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\mathrm{d}\right) \hookrightarrow \left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) $$

of complexes is a quasi-isomorphism.

Proof. The inclusion $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ induces a morphism

$$ \left\{\left({}'F^p\,\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\mathrm{d}\right)\right\}_{p\in\mathbb{Z}} \to \left\{\left({}'F^p\,\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right)\right\}_{p\in\mathbb{Z}} $$

of the associated bounded filtrations, and hence in particular a morphism

$$ \left\{\left({}'E_r^{\bullet,\bullet}\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right),\,{}'\mathrm{d}_r\right)\right\}_{r\in\mathbb{Z}} \to \left\{\left({}'E_r^{\bullet,\bullet}\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right),\,{}'\mathrm{d}_r\right)\right\}_{r\in\mathbb{Z}} $$

of the associated spectral sequences.

By the hypothesis, the inclusion induces an isomorphism at the first level,

$$ \begin{array}{ccc} {}'E_1^{\bullet,\bullet}\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) & \xrightarrow{\simeq} & {}'E_1^{\bullet,\bullet}\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \\ \Downarrow & & \Downarrow \\ H^\bullet\left(\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\mathrm{d}\right) & \longrightarrow & H^\bullet\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) \end{array} $$

and hence, $A^{\bullet,\bullet}$ being bounded, also an isomorphism

$$ H^\bullet\left(\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\mathrm{d}\right) \stackrel{\simeq}{\to} H^\bullet\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) $$

see, e.g., [51, Theorem 3.5]; in particular, the induced map

$$ \left(\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\mathrm{d}\right) \hookrightarrow \left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right) $$

is a quasi-isomorphism. □

1.2. **The Bott-Chern cohomology.** For any $(p,q) \in \mathbb{Z}^2$, other than the cohomologies of $\left(\mathrm{Tot}^\bullet\left(A^{\bullet,\bullet}\right),\,\mathrm{d}\right)$, of $\left(A^{\bullet,q},\,\partial\right)$, and of $\left(A^{p,\bullet},\,\overline{\partial}\right)$, one can consider also the *Bott-Chern cohomology*, [19], namely, the cohomology of the complex

$$ A^{p-1,q-1} \xrightarrow{\partial\overline{\partial}} A^{p,q} \xrightarrow{\partial+\overline{\partial}} A^{p+1,q} \oplus A^{p,q+1} \;, $$

and the *Aeppli cohomology*, [1], namely, the cohomology of the complex

$$ A^{p-1,q} \oplus A^{p,q-1} \xrightarrow{(\partial,\,\overline{\partial})} A^{p,q} \xrightarrow{\partial\overline{\partial}} A^{p+1,q+1} \;. $$

1.2.1. *Conditions yielding a surjective map in Bott-Chern cohomology.* In order to study conditions under which the Bott-Chern cohomology of a double complex can be recovered by means of a suitable sub-complex, we provide the following lemma.

Lemma 1.2. *Let $\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, and let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a sub-complex. Suppose that, for every $p \in \mathbb{Z}$, the induced map $\left(C^{p,\bullet},\,\overline{\partial}\right) \hookrightarrow \left(A^{p,\bullet},\,\overline{\partial}\right)$ of complexes is a quasi-isomorphism. If $\phi \in A^{p,q}$ is such that $\overline{\partial}\phi \in C^{p,q+1}$, then there exist $\tilde{\phi} \in C^{p,q}$ and $\hat{\phi} \in A^{p,q-1}$ such that $\phi = \tilde{\phi} + \overline{\partial}\hat{\phi}$.*

Proof. One has

$$H^{q+1}\left(C^{p,\bullet},\,\overline{\partial}\right) \ni \left(\overline{\partial}\phi \mod \operatorname{im}\overline{\partial}\right) \mapsto \left(0 \mod \operatorname{im}\overline{\partial}\right) \in H^{q+1}\left(A^{p,\bullet},\,\overline{\partial}\right) ;$$

since the map $H^{q+1}\left(C^{p,\bullet},\,\overline{\partial}\right) \stackrel{\simeq}{\to} H^{q+1}\left(A^{p,\bullet},\,\overline{\partial}\right)$ is injective, one gets that $\overline{\partial}\phi \in \operatorname{im}\left(\overline{\partial}\colon C^{p,q} \to C^{p,q+1}\right)$: let $\tilde{\phi}_1 \in C^{p,q}$ be such that

$$\overline{\partial}\phi = \overline{\partial}\tilde{\phi}_1 .$$

Therefore,

$$\left(\left(\phi - \tilde{\phi}_1\right) \mod \operatorname{im}\overline{\partial}\right) \in H^q\left(A^{p,\bullet},\,\overline{\partial}\right) ;$$

since the map $H^q\left(C^{p,\bullet},\,\overline{\partial}\right) \stackrel{\simeq}{\to} H^q\left(A^{p,\bullet},\,\overline{\partial}\right)$ is surjective, one gets that there exist $\tilde{\phi}_2 \in \ker\left(\overline{\partial}\colon C^{p,q} \to C^{p,q+1}\right)$ and $\hat{\phi} \in A^{p,q-1}$ such that

$$\phi - \tilde{\phi}_1 = \tilde{\phi}_2 + \overline{\partial}\hat{\phi} ,$$

that is, $\phi = \tilde{\phi} + \overline{\partial}\hat{\phi}$ where $\tilde{\phi} := \tilde{\phi}_1 + \tilde{\phi}_2 \in C^{p,q}$ and $\hat{\phi} \in A^{p-1,q}$. □

The following result gives a first partial answer concerning the relation between the Bott-Chern cohomology of a double complex and the Bott-Chern cohomology of a suitable sub-complex; compare it with [4, Theorem 3.7], which is in turn inspired by M. Schweitzer's computations on the Iwasawa manifold in [64, §1.c].

Theorem 1.3. *Let $\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, and let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a sub-complex. Fix $(p,q) \in \mathbb{Z}^2$. Suppose that:*

- *(i) for every $r \in \mathbb{Z}$, the induced map $\left(C^{r,\bullet},\,\overline{\partial}\right) \hookrightarrow \left(A^{r,\bullet},\,\overline{\partial}\right)$ of complexes is a quasi-isomorphism,*
- *(ii) for every $s \in \mathbb{Z}$, the induced map $\left(C^{\bullet,s},\,\partial\right) \hookrightarrow \left(A^{\bullet,s},\,\partial\right)$ of complexes is a quasi-isomorphism, and*
- *(iii) the induced map*

$$\frac{\ker\left(\mathrm{d}\colon \operatorname{Tot}^{p+q}\left(C^{\bullet,\bullet}\right) \to \operatorname{Tot}^{p+q+1}\left(C^{\bullet,\bullet}\right)\right) \cap C^{p,q}}{\operatorname{im}\left(\mathrm{d}\colon \operatorname{Tot}^{p+q-1}\left(C^{\bullet,\bullet}\right) \to \operatorname{Tot}^{p+q}\left(C^{\bullet,\bullet}\right)\right)} \to \frac{\ker\left(\mathrm{d}\colon \operatorname{Tot}^{p+q}\left(A^{\bullet,\bullet}\right) \to \operatorname{Tot}^{p+q+1}\left(A^{\bullet,\bullet}\right)\right) \cap A^{p,q}}{\operatorname{im}\left(\mathrm{d}\colon \operatorname{Tot}^{p+q-1}\left(A^{\bullet,\bullet}\right) \to \operatorname{Tot}^{p+q}\left(A^{\bullet,\bullet}\right)\right)}$$

  is surjective.

Then the induced map

$$\left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q} \oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q} \oplus A^{p,q+1}\right)$$

of complexes induces a surjective map in cohomology.

Proof. Up to shifting, assume that $A^{r,s} = \{0\}$ whenever $(r,s) \not\in \mathbb{N}^2$.

`Step 1` – *Firstly, we prove that, under the hypotheses (i) and (ii), the inclusion $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ induces, for every $(r,s) \in \mathbb{Z}^2$, a surjective map*

$$\frac{\operatorname{im}\left(\mathrm{d}\colon \operatorname{Tot}^{r+s-1}\left(C^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(C^{\bullet,\bullet}\right)\right) \cap C^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon C^{r-1,s-1} \to C^{r,s}\right)} \to \frac{\operatorname{im}\left(\mathrm{d}\colon \operatorname{Tot}^{r+s-1}\left(A^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(A^{\bullet,\bullet}\right)\right) \cap A^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)} .$$

Indeed, let

$$\begin{aligned}\left(\omega^{r,s} \mod \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)\right) &:= \left(\mathrm{d}\,\eta \mod \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)\right) \\ &\in \frac{\operatorname{im}\left(\mathrm{d}\colon \operatorname{Tot}^{r+s-1}\left(A^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(A^{\bullet,\bullet}\right)\right) \cap A^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)} .\end{aligned}$$

Consider the bi-degree decomposition $\eta =: \sum_{(a,b)\in\mathbb{Z}^2} \eta^{a,b}$ where $\eta^{a,b} \in A^{a,b}$, for $(a,b) \in \mathbb{Z}^2$. Hence, consider the system

$$\left\{\begin{array}{rcrcll} & & \partial\eta^{r+s-1,0} & = & 0 & \\ \overline{\partial}\eta^{r+s-\ell,\ell-1} & + & \partial\eta^{r+s-\ell-1,\ell} & = & 0 & \text{for} \quad \ell \in \{1,\ldots,s-1\} \\ \overline{\partial}\eta^{r,s-1} & + & \partial\eta^{r-1,s} & = & \omega^{r,s} \mod \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right) & \\ \overline{\partial}\eta^{\ell,r+s-\ell-1} & + & \partial\eta^{\ell-1,r+s-\ell} & = & 0 & \text{for} \quad \ell \in \{1,\ldots,r-1\} \\ \overline{\partial}\eta^{0,r+s-1} & & & = & 0 & \end{array}\right. .$$

Set $\eta^{r+s-2,-1} := 0$, and consider the equation

$$\overline{\partial}\eta^{r+s-\ell,\ell-1} + \partial\eta^{r+s-\ell-1,\ell} \;=\; 0 \quad \text{mod } \operatorname{im}\left(\partial\overline{\partial}\colon A^{r+s-\ell-1,\ell-1} \to A^{r+s-\ell,\ell}\right) \qquad \text{for } \ell \in \{0, \ldots, s-1\} \;.$$

If $\eta^{r+s-\tilde{\ell},\tilde{\ell}-1} \in C^{r+s-\tilde{\ell},\tilde{\ell}-1}$ for some $\tilde{\ell} \in \{0, \ldots, s-1\}$, then, by applying Lemma 1.2 to the double complex $\left(A^{\bullet,\bullet},\, \overline{\partial},\, \partial\right)$, one gets that there exist $\tilde{\eta}^{r+s-\tilde{\ell}-1,\tilde{\ell}} \in C^{r+s-\tilde{\ell}-1,\tilde{\ell}}$ and $\hat{\eta}^{r+s-\tilde{\ell}-2,\tilde{\ell}} \in A^{r+s-\tilde{\ell}-2,\tilde{\ell}}$ such that

$$\eta^{r+s-\tilde{\ell}-1,\tilde{\ell}} \;=\; \tilde{\eta}^{r+s-\tilde{\ell}-1,\tilde{\ell}} + \partial\hat{\eta}^{r+s-\tilde{\ell}-2,\tilde{\ell}} \;;$$

therefore, when $\tilde{\ell} \leq s-2$, one gets the system

$$\begin{cases} \partial\eta^{r+s-1,0} = 0 & \\ \overline{\partial}\eta^{r+s-\ell,\ell-1} + \partial\eta^{r+s-\ell-1,\ell} = 0 & \text{for} \quad \ell \in \{1, \ldots, \tilde{\ell}-1\} \\ \overline{\partial}\eta^{r+s-\tilde{\ell},\tilde{\ell}-1} + \partial\tilde{\eta}^{r+s-\tilde{\ell}-1,\tilde{\ell}} = 0 & \\ \overline{\partial}\tilde{\eta}^{r+s-\tilde{\ell}-1,\tilde{\ell}} + \partial\left(\eta^{r+s-\tilde{\ell}-2,\tilde{\ell}+1} - \overline{\partial}\hat{\eta}^{r+s-\tilde{\ell}-2,\tilde{\ell}}\right) = 0 & \\ \overline{\partial}\left(\eta^{r+s-\tilde{\ell}-2,\tilde{\ell}+1} - \overline{\partial}\hat{\eta}^{r+s-\tilde{\ell}-2,\tilde{\ell}}\right) + \partial\eta^{r+s-\tilde{\ell}-3,\tilde{\ell}+2} = 0 & \\ \overline{\partial}\eta^{r+s-\ell,\ell-1} + \partial\eta^{r+s-\ell-1,\ell} = 0 & \text{for} \quad \ell \in \{\tilde{\ell}+3, \ldots, s-1\} \\ \overline{\partial}\eta^{r,s-1} + \partial\eta^{r-1,s} = \omega^{r,s} \quad \text{mod } \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right) & \\ \overline{\partial}\eta^{\ell,r+s-\ell-1} + \partial\eta^{\ell-1,r+s-\ell} = 0 & \text{for} \quad \ell \in \{1, \ldots, r-1\} \\ \overline{\partial}\eta^{0,r+s-1} = 0 & \end{cases} ,$$

where $\tilde{\eta}^{r+s-\tilde{\ell}-1,\tilde{\ell}} \in C^{r+s-\tilde{\ell}-1,\tilde{\ell}}$, and when $\tilde{\ell} = s-1$, one gets the system

$$\begin{cases} \partial\eta^{r+s-1,0} = 0 & \\ \overline{\partial}\eta^{r+s-\ell,\ell-1} + \partial\eta^{r+s-\ell-1,\ell} = 0 & \text{for} \quad \ell \in \{1, \ldots, s-2\} \\ \overline{\partial}\eta^{r+1,s-2} + \partial\tilde{\eta}^{r,s-1} = 0 & \\ \overline{\partial}\tilde{\eta}^{r,s-1} + \partial\eta^{r-1,s} = \omega^{r,s} \quad \text{mod } \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right) & \\ \overline{\partial}\eta^{\ell,r+s-\ell-1} + \partial\eta^{\ell-1,r+s-\ell} = 0 & \text{for} \quad \ell \in \{1, \ldots, r-1\} \\ \overline{\partial}\eta^{0,r+s-1} = 0 & \end{cases} ,$$

where $\tilde{\eta}^{r,s-1} \in C^{r,s-1}$.

In particular, since $\eta^{r+s-2,-1} = 0 \in C^{r+s-2,-1}$, we may assume that $\eta^{r,s-1} \in C^{r,s-1}$.

Analogously, by applying Lemma 1.2 to the double complex $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$, we may assume that $\eta^{r-1,s} \in C^{r-1,s}$.

Therefore

$$\begin{aligned} \omega^{r,s} \quad \text{mod } \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right) \;&=\; \left(\overline{\partial}\eta^{r,s-1} + \partial\eta^{r-1,s}\right) \quad \text{mod } \operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right) \\ &\in\; \frac{\operatorname{im}\left(\mathrm{d}\colon\ \operatorname{Tot}^{r+s-1}\left(C^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(C^{\bullet,\bullet}\right)\right) \cap C^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)} \;, \end{aligned}$$

that is, the induced map

$$\frac{\operatorname{im}\left(\mathrm{d}\colon\ \operatorname{Tot}^{r+s-1}\left(C^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(C^{\bullet,\bullet}\right)\right) \cap C^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon C^{r-1,s-1} \to C^{r,s}\right)} \to \frac{\operatorname{im}\left(\mathrm{d}\colon\ \operatorname{Tot}^{r+s-1}\left(A^{\bullet,\bullet}\right) \to \operatorname{Tot}^{r+s}\left(A^{\bullet,\bullet}\right)\right) \cap A^{r,s}}{\operatorname{im}\left(\partial\overline{\partial}\colon A^{r-1,s-1} \to A^{r,s}\right)}$$

is surjective.

`Step 2` – *Now, we prove that, under the additional assumption (iii), the induced map*

$$\frac{\ker\left(\partial\colon C^{p,q} \to C^{p+1,q}\right) \cap \ker\left(\overline{\partial}\colon C^{p,q} \to C^{p,q+1}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon C^{p-1,q-1} \to C^{p,q}\right)} \to \frac{\ker\left(\partial\colon A^{p,q} \to A^{p+1,q}\right) \cap \ker\left(\overline{\partial}\colon A^{p,q} \to A^{p,q+1}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon A^{p-1,q-1} \to A^{p,q}\right)}$$

is surjective.
Indeed, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & \\
\frac{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(C^{\bullet,\bullet})\to\mathrm{Tot}^{p+q}(C^{\bullet,\bullet})\right)\cap C^{p,q}}{\mathrm{im}\left(\partial\overline{\partial}\colon C^{p-1,q-1}\to C^{p,q}\right)} \arrow[r] \arrow[d] & \frac{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(A^{\bullet,\bullet})\to\mathrm{Tot}^{p+q}(A^{\bullet,\bullet})\right)\cap A^{p,q}}{\mathrm{im}\left(\partial\overline{\partial}\colon A^{p-1,q-1}\to A^{p,q}\right)} \arrow[r] \arrow[d] & 0 \\
\frac{\ker\left(\partial\colon C^{p,q}\to C^{p+1,q}\right)\cap\ker\left(\overline{\partial}\colon C^{p,q}\to C^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon C^{p-1,q-1}\to C^{p,q}\right)} \arrow[r] \arrow[d] & \frac{\ker\left(\partial\colon A^{p,q}\to A^{p+1,q}\right)\cap\ker\left(\overline{\partial}\colon A^{p,q}\to A^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon A^{p-1,q-1}\to A^{p,q}\right)} \arrow[d] & \\
\frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(C^{\bullet,\bullet})\to\mathrm{Tot}^{p+q+1}(C^{\bullet,\bullet})\right)\cap C^{p,q}}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(C^{\bullet,\bullet})\to\mathrm{Tot}^{p+q}(C^{\bullet,\bullet})\right)} \arrow[r] \arrow[d] & \frac{\ker\left(\mathrm{d}\colon \mathrm{Tot}^{p+q}(A^{\bullet,\bullet})\to\mathrm{Tot}^{p+q+1}(A^{\bullet,\bullet})\right)\cap A^{p,q}}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{Tot}^{p+q-1}(A^{\bullet,\bullet})\to\mathrm{Tot}^{p+q}(A^{\bullet,\bullet})\right)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, equal] & 0 &
\end{tikzcd}$$

whose rows and columns are exact. By the Five Lemma, see, e.g., [51, page 26], the map

$$\frac{\ker\left(\partial\colon C^{p,q}\to C^{p+1,q}\right)\cap\ker\left(\overline{\partial}\colon C^{p,q}\to C^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon C^{p-1,q-1}\to C^{p,q}\right)} \to \frac{\ker\left(\partial\colon A^{p,q}\to A^{p+1,q}\right)\cap\ker\left(\overline{\partial}\colon A^{p,q}\to A^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon A^{p-1,q-1}\to A^{p,q}\right)}$$

is surjective, completing the proof. □

1.2.2. *Conditions yielding an injective map in Bott-Chern cohomology.* In order to provide conditions under which the inclusion of a suitable sub-complex induces an injective map in Bott-Chern cohomology, we consider a further structure of Hilbert space on the double complex. (For similar results in the case of solvmanifolds, see [22, Lemma 9], [4, Lemma 3.6].)

Let A be a Hilbert space, with inner product $\langle\cdot\,|\,\cdot\cdot\rangle\colon A\times A\to\mathbb{C}$. Denote by $\|\cdot\|:=\langle\cdot\,|\,\cdot\rangle^{1/2}$ the associated norm.

Given a densely-defined linear operator $L\colon A\supseteq\mathrm{dom}(L)\to A$ on A, denote by

$$L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\colon\ \mathrm{dom}\left(L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)\to A$$

its $\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint operator, that is, the unique linear operator with domain

$$\mathrm{dom}\left(L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)\ :=\ \{y\in A\ :\ \langle L\,\cdot\,|\,y\rangle\colon\ \mathrm{dom}(L)\to\mathbb{C}\text{ is continuous}\}$$

and defined by

$$\forall x\in\mathrm{dom}(L),\ \forall y\in\mathrm{dom}\left(L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\right),\qquad \langle Lx\,|\,y\rangle\ =\ \left\langle x\,\middle|\,L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}y\right\rangle\ .$$

Given a closed sub-space C of A, denote the induced inner product on C by $\langle\cdot\,|\,\cdot\cdot\rangle_C:=\langle\cdot\,|\,\cdot\cdot\rangle\lfloor_{C\times C}\colon C\times C\to\mathbb{C}$, and the orthogonal projection onto C by $\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\colon A\to C\subseteq A$. One has that

$$\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_C=\ \mathrm{id}_C\qquad\text{and}\qquad\left\langle C\,\middle|\,\left(\mathrm{id}_A-\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)(A)\right\rangle\ =\ \{0\}\ .$$

(To simplify notations, we do not specify the inner product $\langle\cdot\,|\,\cdot\cdot\rangle$ in writing the projection or the adjoint, whenever it is clear from the context.)

We firstly record the following lemma, stating that, if L commutes with π^C, then also L^* does.

Lemma 1.4. *Let A be a Hilbert space, with inner product $\langle\cdot\,|\,\cdot\cdot\rangle$. Let $L\colon A\supseteq\mathrm{dom}(L)\to A$ be a densely-defined linear operator on A. Let C be a closed sub-space of A contained in $\mathrm{dom}(L)$ and in $\mathrm{dom}\left(L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)$. Suppose that*

$$\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\circ L\ =\ L\circ\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\colon\ \mathrm{dom}(L)\to C\ .$$

Then

$$\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\circ L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\ =\ L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\circ\pi^C_{\langle\cdot\,|\,\cdot\cdot\rangle}\colon\ \mathrm{dom}\left(L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)\to C\ ;$$

in particular, $L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_C\colon C\to C$, and hence $(L\lfloor_C)^*_{\langle\cdot\,|\,\cdot\cdot\rangle_C}=L^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_C$.

Proof. It suffices to note that $\pi^C \colon A \to C \subseteq A$ is self-$\langle \cdot \,|\, \cdot\cdot \rangle$-adjoint: for any $\alpha, \beta \in A$,

$$\left\langle \pi^C \alpha \,\middle|\, \beta \right\rangle \;=\; \left\langle \pi^C \alpha \,\middle|\, \beta - \left(\beta - \pi^C \beta\right) \right\rangle \;=\; \left\langle \pi^C \alpha \,\middle|\, \pi^C \beta \right\rangle \;=\; \left\langle \pi^C \alpha + \left(\alpha - \pi^C \alpha\right) \,\middle|\, \pi^C \beta \right\rangle \;=\; \left\langle \alpha \,\middle|\, \pi^C \beta \right\rangle \;.$$

It follows straightforwardly that $\pi^C \circ L^* = L^* \circ \pi^C \colon \operatorname{dom}(L^*) \to C$. In particular, since $\pi^C \lfloor_C = \mathrm{id}_C$ and $C \subseteq \operatorname{dom}(L^*)$, it follows that $L^*(C) = \left(L^* \circ \pi^C\right)(C) = \left(\pi^C \circ L^*\right)(C) \subseteq C$, and hence $L^* \lfloor_C = (L\lfloor_C)^*_{\langle \cdot | \cdot\cdot \rangle_C} \colon C \to C$. $\square$

Now, let $A^{\bullet,\bullet}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle \cdot \,|\, \cdot\cdot \rangle$ such that $\left\langle A^{p,q} \,\middle|\, A^{p',q'} \right\rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let

$$\partial \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}(\partial)^{\bullet,\bullet} \to A^{\bullet+1,\bullet} \qquad \text{and} \qquad \overline{\partial} \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}(\overline{\partial})^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$$

be densely-defined linear operators yielding a structure $\left(\left(\operatorname{dom}(\partial) \cap \operatorname{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Denote by

$$\partial^* \;:=\; \partial^*_{\langle \cdot | \cdot\cdot \rangle} \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}(\partial^*)^{\bullet,\bullet} \to A^{\bullet-1,\bullet} \qquad \text{and} \qquad \overline{\partial}^* \;:=\; \overline{\partial}^*_{\langle \cdot | \cdot\cdot \rangle} \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}\left(\overline{\partial}^*\right)^{\bullet,\bullet} \to A^{\bullet,\bullet-1}$$

the $\langle \cdot \,|\, \cdot\cdot \rangle$-adjoint operators of ∂ and, respectively, $\overline{\partial}$.

Following [46, Proposition 5], see also [64, §2.b, §2.c], define the (densely-defined) self-$\langle \cdot \,|\, \cdot\cdot \rangle$-adjoint operator

$$\begin{aligned} \tilde{\Delta}^{BC} \;:=\; \tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle} \;&:=\; \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \\ &\in\; \operatorname{Hom}^{0,0}\left(\operatorname{dom}\left(\tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle}\right)^{\bullet,\bullet};\, A^{\bullet,\bullet}\right) \;. \end{aligned}$$

The following lemma states that, under a suitable decomposition hypothesis, the Bott-Chern cohomology of $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ is isomorphic to $\ker \tilde{\Delta}^{BC}$.

Lemma 1.5. *Let $A^{\bullet,\bullet}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle \cdot \,|\, \cdot\cdot \rangle$ such that $\left\langle A^{p,q} \,\middle|\, A^{p',q'} \right\rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let $\partial \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}(\partial)^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$ and $\overline{\partial} \colon A^{\bullet,\bullet} \supseteq \operatorname{dom}(\overline{\partial})^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$ be densely-defined linear operators yielding a structure $\left(\left(\operatorname{dom}(\partial) \cap \operatorname{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Suppose that the operator $\tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle} \in \operatorname{Hom}^{0,0}\left(\operatorname{dom}\left(\tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle}\right)^{\bullet,\bullet};\, A^{\bullet,\bullet}\right)$ induces the decomposition*

$$\operatorname{dom}\left(\tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle}\right) \;=\; \ker \tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle} \oplus \operatorname{im} \tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle} \;.$$

Then, for every $(p,q) \in \mathbb{Z}^2$, the induced map

$$\left(0 \to \ker \tilde{\Delta}^{BC}_{\langle \cdot | \cdot\cdot \rangle} \cap A^{p,q} \to 0\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q} \oplus A^{p,q+1}\right)$$

is a quasi-isomorphism.

Proof. Note that, for every $\eta \in \operatorname{dom}\left(\tilde{\Delta}^{BC}\right)$, one has

$$\left\langle \tilde{\Delta}^{BC} \eta \,\middle|\, \eta \right\rangle \;=\; \left\| \left(\partial\overline{\partial}\right)^* \eta \right\|^2 + \left\| \partial\overline{\partial}\eta \right\|^2 + \left\| \partial^*\overline{\partial}\eta \right\|^2 + \left\| \overline{\partial}^*\partial\eta \right\|^2 + \left\| \overline{\partial}\eta \right\|^2 + \left\| \partial\eta \right\|^2 \;,$$

hence

$$\ker \tilde{\Delta}^{BC} \;=\; \ker \partial \cap \ker \overline{\partial} \cap \ker \left(\partial\overline{\partial}\right)^* \;.$$

On the other hand, since $\operatorname{im} \tilde{\Delta}^{BC} \subseteq \operatorname{im} \partial\overline{\partial} \oplus \left(\operatorname{im} \partial^* + \operatorname{im} \overline{\partial}^*\right)$ and $\left(\operatorname{im} \partial^* + \operatorname{im} \overline{\partial}^*\right) \cap \left(\ker \partial \cap \ker \overline{\partial}\right) = \{0\}$, one has

$$\operatorname{im} \tilde{\Delta}^{BC} \cap \left(\ker \partial \cap \ker \overline{\partial}\right) \;=\; \operatorname{im} \partial\overline{\partial} \;.$$

It follows that

$$\ker \tilde{\Delta}^{BC} \cap A^{p,q} \stackrel{\simeq}{\to} \frac{\ker \tilde{\Delta}^{BC} \cap A^{p,q} + \operatorname{im} \partial\overline{\partial} \cap A^{p,q}}{\operatorname{im}\left(\partial\overline{\partial} \colon A^{p-1,q-1} \to A^{p,q}\right)} \;\simeq\; \frac{\ker\left(\partial + \overline{\partial} \colon A^{p,q} \to A^{p+1,q} \oplus A^{p,q+1}\right)}{\operatorname{im}\left(\partial\overline{\partial} \colon A^{p-1,q-1} \to A^{p,q}\right)} \;,$$

completing the proof. $\square$

We have now the following result.

Theorem 1.6. *Let $A^{\bullet,\bullet}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle\cdot\,|\,\cdot\cdot\rangle$ such that $\left\langle A^{p,q} \,\middle|\, A^{p',q'}\right\rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let $\partial\colon A^{\bullet,\bullet} \supseteq \mathrm{dom}(\partial)^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$ and $\overline{\partial}\colon A^{\bullet,\bullet} \supseteq \mathrm{dom}(\overline{\partial})^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$ be densely-defined linear operators yielding a structure $\left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Let*

$$j\colon\; (C^{\bullet,\bullet},\, \partial,\, \overline{\partial}) \hookrightarrow \left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$$

be a sub-complex. Suppose that:

(i) the operator $\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \in \mathrm{Hom}^{0,0}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right)^{\bullet,\bullet};\, A^{\bullet,\bullet}\right)$ induces the decomposition

$$\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right) \;=\; \ker\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \oplus \mathrm{im}\,\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\;;$$

(ii) it holds that

$$\partial^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \left(\partial\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon\, \mathrm{dom}\left(\partial^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}}\right)^{\bullet,\bullet} \to C^{\bullet-1,\bullet}$$

and

$$\overline{\partial}^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \left(\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon\, \mathrm{dom}\left(\overline{\partial}^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}}\right)^{\bullet,\bullet} \to C^{\bullet,\bullet-1}\;;$$

in particular, it follows that

$$\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \;\in\; \mathrm{Hom}^{0,0}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}}\right)^{\bullet,\bullet};\, C^{\bullet,\bullet}\right)\;;$$

(iii) the operator $\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \in \mathrm{Hom}^{0,0}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}}\right)^{\bullet,\bullet};\, C^{\bullet,\bullet}\right)$ induces the decomposition

$$\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}}\right) \;=\; \ker\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \oplus \mathrm{im}\,\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}}\;.$$

Then, for every $(p,q) \in \mathbb{Z}^2$, the induced map

$$j\colon\; \left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q} \oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q} \oplus A^{p,q+1}\right)$$

of complexes induces an injective map j^* in cohomology.

Proof. By Lemma 1.5 and under the hypotheses *(i)*, *(ii)*, and *(iii)*, one gets that both

$$\left(0 \to \ker\tilde{\Delta}^{BC} \cap A^{p,q} \to 0\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q} \oplus A^{p,q+1}\right)$$

and

$$\left(0 \to \ker\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}} \cap C^{p,q} = \ker\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \cap C^{p,q} \to 0\right) \hookrightarrow \left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q} \oplus C^{p,q+1}\right)$$

are quasi-isomorphisms.

Hence, one has the commutative diagram

$$\begin{array}{ccc}
\ker\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}} \cap C^{p,q} & \xrightarrow{\simeq} & \dfrac{\ker\left(\partial+\overline{\partial}\colon C^{p,q}\to C^{p+1,q}\oplus C^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon C^{p-1,q-1}\to C^{p,q}\right)} \\
\big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle j^*} \\
\ker\tilde{\Delta}^{BC} \cap A^{p,q} & \xrightarrow{\simeq} & \dfrac{\ker\left(\partial+\overline{\partial}\colon A^{p,q}\to A^{p+1,q}\oplus A^{p,q+1}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon A^{p-1,q-1}\to A^{p,q}\right)}
\end{array}$$

getting that j^* is injective. □

By using Lemma 1.4, one gets the following corollary of Theorem 1.6, concerning closed sub-complexes.

Corollary 1.7. *Let $A^{\bullet,\bullet}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle\cdot\,|\,\cdot\cdot\rangle$ such that $\left\langle A^{p,q} \,\middle|\, A^{p',q'}\right\rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let $\partial\colon A^{\bullet,\bullet} \supseteq \mathrm{dom}(\partial)^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$ and $\overline{\partial}\colon A^{\bullet,\bullet} \supseteq \mathrm{dom}(\overline{\partial})^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$ be densely-defined linear operators yielding a structure $\left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Let $j\colon\; (C^{\bullet,\bullet},\, \partial,\, \overline{\partial}) \hookrightarrow \left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a closed sub-complex. Suppose that:*

(i) *the operator* $\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \in \mathrm{Hom}^{0,0}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right)^{\bullet,\bullet}; A^{\bullet,\bullet}\right)$ *induces the decomposition*

$$\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right) = \ker\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \oplus \mathrm{im}\,\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle};$$

(ii) $C^{\bullet,\bullet} \subseteq \mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\cap\mathrm{dom}\left(\partial^*_{\langle\cdot|\cdot\cdot\rangle}\right)\cap\mathrm{dom}\left(\overline{\partial}^*_{\langle\cdot|\cdot\cdot\rangle}\right)$, *and* $\pi^{C^{\bullet,\bullet}}\circ\partial = \partial\circ\pi^{C^{\bullet,\bullet}} \colon \mathrm{dom}(\partial)^{\bullet,\bullet} \to C^{\bullet+1,\bullet}$ *and* $\pi^{C^{\bullet,\bullet}}\circ\overline{\partial} = \overline{\partial}\circ\pi^{C^{\bullet,\bullet}} \colon \mathrm{dom}(\overline{\partial})^{\bullet,\bullet} \to C^{\bullet,\bullet+1}$.

Then, for every $(p,q)\in\mathbb{Z}^2$, *the induced map*

$$j\colon \left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q}\oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q}\oplus A^{p,q+1}\right)$$

of complexes induces an injective map j^* *in cohomology.*

Proof. By Lemma 1.4, one has $\pi^{C^{\bullet,\bullet}}\circ\partial^* = \partial^*\circ\pi^{C^{\bullet,\bullet}} \colon \mathrm{dom}\,(\partial^*)^{\bullet,\bullet} \to C^{\bullet-1,\bullet}$ and $\pi^{C^{\bullet,\bullet}}\circ\overline{\partial}^* = \overline{\partial}^*\circ\pi^{C^{\bullet,\bullet}} \colon \mathrm{dom}\left(\overline{\partial}^*\right)^{\bullet,\bullet} \to C^{\bullet,\bullet-1}$, and hence in particular $\partial^*\lfloor_{C^{\bullet,\bullet}} = (\partial\lfloor_{C^{\bullet,\bullet}})^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet-1,\bullet}$ and $\overline{\partial}^*\lfloor_{C^{\bullet,\bullet}} = (\overline{\partial}\lfloor_{C^{\bullet,\bullet}})^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet,\bullet-1}$.

Furthermore, it follows that $\pi^{C^{\bullet,\bullet}}\circ\tilde{\Delta}^{BC} = \tilde{\Delta}^{BC}\circ\pi^{C^{\bullet,\bullet}} \colon \mathrm{dom}\left(\tilde{\Delta}^{BC}\right)^{\bullet,\bullet} \to C^{\bullet,\bullet}$. In particular, it follows that

$$\pi^{C^{\bullet,\bullet}}\left(\ker\tilde{\Delta}^{BC}\right) = \ker\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}} \qquad\text{and}\qquad \pi^{C^{\bullet,\bullet}}\left(\mathrm{im}\,\tilde{\Delta}^{BC}\right) = \mathrm{im}\,\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}},$$

and hence one gets the decomposition

$$\begin{aligned}\mathrm{dom}\left(\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}}\right)^{\bullet,\bullet} &= \pi^{C^{\bullet,\bullet}}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}\right)^{\bullet,\bullet}\right) = \pi^{C^{\bullet,\bullet}}\left(\ker\tilde{\Delta}^{BC}\right) + \pi^{C^{\bullet,\bullet}}\left(\mathrm{im}\,\tilde{\Delta}^{BC}\right)\\ &= \ker\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}}\oplus\mathrm{im}\,\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}}.\end{aligned}$$

Hence the hypotheses of Theorem 1.6 are satisfied, completing the proof. □

Note that hypothesis *(iii)* in Theorem 1.6 is satisfied whenever the sub-complex $C^{\bullet,\bullet}$ is finite-dimensional.

Corollary 1.8. *Let* $A^{\bullet,\bullet}$ *be a bounded* $\mathbb{Z}^2$*-graded vector space with a structure of Hilbert space, with inner product* $\langle\cdot|\cdot\cdot\rangle$ *such that* $\left\langle A^{p,q}\,\middle|\,A^{p',q'}\right\rangle = \{0\}$ *for every* $(p,q)\neq(p',q')$. *Let* $\partial\colon A^{\bullet,\bullet}\supseteq\mathrm{dom}(\partial)^{\bullet,\bullet}\to A^{\bullet+1,\bullet}$ *and* $\overline{\partial}\colon A^{\bullet,\bullet}\supseteq\mathrm{dom}(\overline{\partial})^{\bullet,\bullet}\to A^{\bullet,\bullet+1}$ *be densely-defined linear operators yielding a structure* $\left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\partial,\overline{\partial}\right)$ *of bounded double complex of* $\mathbb{C}$*-vector spaces. Let* $j\colon\left(C^{\bullet,\bullet},\partial,\overline{\partial}\right)\hookrightarrow\left(\left(\mathrm{dom}(\partial)\cap\mathrm{dom}(\overline{\partial})\right)^{\bullet,\bullet},\partial,\overline{\partial}\right)$ *be a sub-complex. Suppose that:*

(i) *the operator* $\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \in \mathrm{Hom}^{0,0}\left(\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right)^{\bullet,\bullet}; A^{\bullet,\bullet}\right)$ *induces the decomposition*

$$\mathrm{dom}\left(\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle}\right)^{\bullet,\bullet} = \ker\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle} \oplus \mathrm{im}\,\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle};$$

(ii) $C^{\bullet,\bullet}$ *is finite-dimensional;*

(iii) *it holds that*

$$\partial^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} = (\partial\lfloor_{C^{\bullet,\bullet}})^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet-1,\bullet}$$

and

$$\overline{\partial}^*_{\langle\cdot|\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} = \left(\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet,\bullet-1}.$$

Then, for every $(p,q)\in\mathbb{Z}^2$, *the induced map*

$$j\colon \left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q}\oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q}\oplus A^{p,q+1}\right)$$

of complexes induces an injective map j^* *in cohomology.*

Proof. Note that, if $C^{\bullet,\bullet}\subseteq\left(\mathrm{dom}\,\partial\cap\mathrm{dom}\,\overline{\partial}\right)^{\bullet,\bullet}$ is finite-dimensional, as in *(ii)*, then the $\mathbb{C}$-linear operators $\partial\lfloor_{C^{\bullet,\bullet}}\colon C^{\bullet,\bullet}\to C^{\bullet+1,\bullet}$ and $\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\colon C^{\bullet,\bullet}\to C^{\bullet,\bullet+1}$ are continuous, and hence $\mathrm{dom}\,(\partial\lfloor_{C^{\bullet,\bullet}})^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} = \mathrm{dom}\,(\partial^*\lfloor_{C^{\bullet,\bullet}}) = C^{\bullet,\bullet}$ and $\mathrm{dom}\left(\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} = \mathrm{dom}\left(\overline{\partial}^*\lfloor_{C^{\bullet,\bullet}}\right) = C^{\bullet,\bullet}$. By hypothesis *(iii)*, it follows that $\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}} = \tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\in\mathrm{End}^{0,0}\,(C^{\bullet,\bullet})$. In particular, $\mathrm{dom}\,\tilde{\Delta}^{BC}_{\langle\cdot|\cdot\cdot\rangle_{C^{\bullet,\bullet}}} = \mathrm{dom}\,\tilde{\Delta}^{BC}\lfloor_{C^{\bullet,\bullet}} = C^{\bullet,\bullet}$.

Hence, in order to apply Theorem 1.6, it suffices to show that, given a finite-dimensional $\mathbb{C}$-vector space C endowed with an inner product $\langle\cdot\,|\,\cdot\cdot\rangle$, any self-$\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint endomorphism $L \in \mathrm{Hom}(C)$ yields a decomposition

$$C \;=\; \ker L \oplus \operatorname{im} L\;.$$

Indeed, take $\ker L \subseteq C$ and let $V \subseteq C$ be the $\mathbb{C}$-vector sub-space of C being $\langle\cdot\,|\,\cdot\cdot\rangle$-orthogonal to $\ker L$; in particular, $C = \ker L \overset{\perp}{\oplus} V$. It suffices to show that $V = \operatorname{im} L$. Since L is self-$\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint, then $\langle \operatorname{im} L \,|\, \ker L\rangle = \{0\}$, and hence $\operatorname{im} L \subseteq V$. Since $\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} \operatorname{im} L + \dim_{\mathbb{C}} \ker L < +\infty$, it follows that $V = \operatorname{im} L$. $\square$

Remark 1.9. Obviously, Theorem 1.6, as well as its corollaries, holds, with straightforward modifications, also for the cohomologies associated to the operators $\Delta_{\langle\cdot\,|\,\cdot\cdot\rangle} := [\mathrm{d},\, \mathrm{d}^*]$, and $\square_{\langle\cdot\,|\,\cdot\cdot\rangle} := [\partial,\, \partial^*]$, and $\overline{\square}_{\langle\cdot\,|\,\cdot\cdot\rangle} := \left[\overline{\partial},\, \overline{\partial}^*\right]$, and $\tilde\Delta^{A}_{\langle\cdot\,|\,\cdot\cdot\rangle} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^*$.

2. Applications

We are now interested in applying the general results of the previous section to suitable sub-complexes of the double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$, where X is a compact complex manifold. We are especially interested in the case when X is a solvmanifold.

2.1. Complexes of PD-type.

Let $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a double complex of $\mathbb{C}$-vector spaces. Suppose that $A^{\bullet,\bullet}$ have a structure $\wedge$ of $\mathbb{C}$-algebra being compatible with the $\mathbb{Z}^2$-grading (namely, $A^{p,q} \wedge A^{p',q'} \subseteq A^{p+p',q+q'}$ for every $(p,q), (p',q') \in \mathbb{Z}^2$), and with respect to which $\mathrm{d} := \partial + \overline{\partial}$ satisfies the Leibniz rule, namely,

$$\text{for every } a \in \mathrm{Tot}^{\hat a} A^{\bullet,\bullet}\;, \qquad [\mathrm{d},\, a \wedge \cdot] \;=\; \mathrm{d}\, a \wedge \cdot \;\in\; \mathrm{End}^{\hat a+1}\left(\mathrm{Tot}^{\bullet} A^{\bullet,\bullet}\right)\;.$$

Following the notation introduced in [44, §2] by the second author, $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ is said to be a *bi-differential $\mathbb{Z}^2$-graded algebra of* PD*-type* if

(i) whenever $p < 0$ or $q < 0$, then $A^{p,q} = \{0\}$, and $A^{0,0} = \mathbb{C}\,\langle 1\rangle$;
(ii) there exists $n \in \mathbb{N}$ such that, whenever $p > n$ or $q > n$, then $A^{p,q} = \{0\}$, and $A^{n,n} = \mathbb{C}\,\langle v\rangle$; (call n the PD*-dimension of* $A^{\bullet,\bullet}$;)
(iii) for every $(h,k) \in \{0,\ldots,n\}^2$, the bi-$\mathbb{C}$-linear map $A^{h,k} \times A^{n-h,n-k} \to A^{n,n} \stackrel{\simeq}{\to} \mathbb{C}$ induced by $\wedge$ is non-degenerate;
(iv) $\mathrm{d}\,\mathrm{Tot}^0 A^{\bullet,\bullet} = \{0\}$ and $\mathrm{d}\,\mathrm{Tot}^{2n-1} A^{\bullet,\bullet} = \{0\}$.

Given a bi-differential $\mathbb{Z}^2$-graded algebra $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ of PD-type, let $\langle\cdot\,|\,\cdot\cdot\rangle$ be an inner product on $A^{\bullet,\bullet}$ being compatible with the $\mathbb{Z}^2$-grading, namely, $\left\langle A^{p,q} \,\middle|\, A^{p',q'}\right\rangle = \{0\}$ whenever $(p,q) \neq (p',q')$, and being compatible with the PD-type structure, namely, $\langle v\,|\,v\rangle = 1$. Define the $\mathbb{C}$-anti-linear map

$$\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle} \colon A^{\bullet_1,\bullet_2} \to A^{n-\bullet_1,n-\bullet_2} \qquad \text{such that} \qquad \text{for every } \alpha,\beta \in A^{\bullet,\bullet}\;, \quad \alpha \wedge \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\beta \;=\; \langle\alpha\,|\,\beta\rangle \cdot v$$

(as above, we will understand the scalar product $\langle\cdot\,|\,\cdot\cdot\rangle$ whenever it is clear from the context).

By considering the Hilbert space given by the $\langle\cdot\,|\,\cdot\cdot\rangle$-completion of $A^{\bullet,\bullet}$, one has that the operators

$$\partial^* \;:=\; -\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\,\partial\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle} \colon A^{\bullet,\bullet} \to A^{\bullet-1,\bullet} \qquad \text{and} \qquad \overline{\partial}^* \;:=\; -\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\,\overline{\partial}\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle} \colon A^{\bullet,\bullet} \to A^{\bullet,\bullet-1}$$

are in fact the $\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint operators $\partial^*_{\langle\cdot\,|\,\cdot\cdot\rangle}$, respectively $\overline{\partial}^*_{\langle\cdot\,|\,\cdot\cdot\rangle}$, of $\partial \colon A^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$, respectively $\overline{\partial} \colon A^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$, and the operator

$$\mathrm{d}^* \;:=\; -\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\;\mathrm{d}\;\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle} \;=\; \partial^* + \overline{\partial}^* \colon\; \mathrm{Tot}^{\bullet} A^{\bullet,\bullet} \to \mathrm{Tot}^{\bullet-1} A^{\bullet,\bullet}$$

is in fact the $\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint operator $\mathrm{d}^*_{\langle\cdot\,|\,\cdot\cdot\rangle}$ of $\mathrm{d} := \partial + \overline{\partial} \colon\; \mathrm{Tot}^{\bullet} A^{\bullet,\bullet} \to \mathrm{Tot}^{\bullet+1} A^{\bullet,\bullet}$, [44, Lemma 2.4].

The following result is an application of Corollary 1.8 to the case of bi-differential $\mathbb{Z}^2$-graded algebras of PD-type.

Proposition 2.1. *Let $\left(A^{\bullet,\bullet},\, \partial,\, \overline{\partial}\right)$ be a bi-differential $\mathbb{Z}^2$-graded algebra of* PD*-type of* PD*-dimension n. Let $\langle\cdot\,|\,\cdot\cdot\rangle$ be an inner product on $A^{\bullet,\bullet}$ being compatible with the $\mathbb{Z}^2$-grading and with the* PD*-type structure. Consider the Hilbert space given by the $\langle\cdot\,|\,\cdot\cdot\rangle$-completion of $A^{\bullet,\bullet}$, and suppose that the operator $\tilde\Delta^{BC}_{\langle\cdot\,|\,\cdot\cdot\rangle} \in \mathrm{End}^{0,0}\left(A^{\bullet,\bullet}\right)$ induces the decomposition*

$$A^{\bullet,\bullet} \;=\; \ker\tilde\Delta^{BC}_{\langle\cdot\,|\,\cdot\cdot\rangle} \oplus \operatorname{im}\tilde\Delta^{BC}_{\langle\cdot\,|\,\cdot\cdot\rangle}\;.$$

Let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a finite-dimensional sub-complex of $\left(A^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ having a structure of bi-differential $\mathbb{Z}^2$-graded algebra of PD*-type of* PD*-dimension n induced by $A^{\bullet,\bullet}$. Suppose that*

$$\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \colon C^{\bullet,\bullet} \to C^{n-\bullet,n-\bullet} \;.$$

Then, for any $(p,q)\in\mathbb{Z}^2$, the induced inclusions

$$\left(\mathrm{Tot}^\bullet\left(C^{\bullet,\bullet}\right),\,\partial+\overline{\partial}\right) \hookrightarrow \left(\mathrm{Tot}^\bullet A^{\bullet,\bullet},\,\partial+\overline{\partial}\right) \;, \qquad \left(C^{\bullet,q},\,\partial\right) \hookrightarrow \left(A^{\bullet,q},\,\partial\right) \;, \qquad \left(C^{p,\bullet},\,\overline{\partial}\right) \hookrightarrow \left(A^{p,\bullet},\,\overline{\partial}\right) \;,$$

and

$$\left(C^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} C^{p,q} \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q}\oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \stackrel{\partial\overline{\partial}}{\to} A^{p,q} \stackrel{\partial+\overline{\partial}}{\to} A^{p+1,q}\oplus A^{p,q+1}\right)$$

and

$$\left(C^{p-1,q}\oplus C^{p,q-1} \stackrel{(\partial,\overline{\partial})}{\to} C^{p,q} \stackrel{\partial\overline{\partial}}{\to} C^{p+1,q+1}\right) \hookrightarrow \left(A^{p-1,q}\oplus A^{p,q-1} \stackrel{(\partial,\overline{\partial})}{\to} A^{p,q} \stackrel{\partial\overline{\partial}}{\to} A^{p+1,q+1}\right)$$

induce injective maps in cohomology.

Proof. By the hypothesis that $\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \colon C^{\bullet,\bullet} \to C^{n-\bullet,n-\bullet}$, one gets that

$$\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}$$

(indeed, let $\alpha\in C^{\bullet,\bullet}$; then, for any $\beta\in C^{\bullet,\bullet}$, it holds that $\left(\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\alpha - \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\alpha\right)\wedge\beta = 0$; by taking $\beta = \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\left(\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\alpha - \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\alpha\right) \in C^{\bullet,\bullet}$, one gets hence that $\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\alpha = \bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\alpha$). In particular, it follows that

$$\partial^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \left(-\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\,\partial\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)\lfloor_{C^{\bullet,\bullet}} \;=\; -\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\,\partial\lfloor_{C^{\bullet,\bullet}}\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \;=\; \left(\partial\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet-1,\bullet}$$

and

$$\overline{\partial}^*_{\langle\cdot\,|\,\cdot\cdot\rangle}\lfloor_{C^{\bullet,\bullet}} \;=\; \left(-\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\,\overline{\partial}\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle}\right)\lfloor_{C^{\bullet,\bullet}} \;=\; -\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}}\,\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\,\bar{*}_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \;=\; \left(\overline{\partial}\lfloor_{C^{\bullet,\bullet}}\right)^*_{\langle\cdot\,|\,\cdot\cdot\rangle_{C^{\bullet,\bullet}}} \colon C^{\bullet,\bullet}\to C^{\bullet,\bullet-1} \;.$$

Hence Corollary 1.8, see also Remark 1.9, applies. □

2.2. **Compact complex manifolds.** Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g. (Note that all manifolds are assumed to have no boundary.)

By considering the ($\mathbb{C}$-anti-linear) Hodge-$*$-operator

$$\bar{*}_g\colon\; \wedge^{\bullet_1,\bullet_2}X \to \wedge^{n-\bullet_1,n-\bullet_2}X$$

and the inner product

$$\langle\cdot\,|\,\cdot\cdot\rangle \;:=\; \int_X \cdot\wedge\bar{*}_g(\cdot\cdot) \;,$$

one gets that the double complex $\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ has a structure of bi-differential $\mathbb{Z}^2$-graded algebra of PD-type of PD-dimension n, such that $\langle\cdot\,|\,\cdot\cdot\rangle$ is compatible with the $\mathbb{Z}^2$-grading and with the PD-type structure of $\wedge^{\bullet,\bullet}X$.

The 2nd order self-$\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint elliptic differential operators

$$\Delta_g \;:=\; [\mathrm{d},\,\mathrm{d}^*] \;\in\; \mathrm{End}^0\left(\wedge^\bullet X\otimes\mathbb{C}\right) \;,$$

and

$$\square_g \;:=\; [\partial,\,\partial^*] \;\in\; \mathrm{End}^{0,0}\left(\wedge^{\bullet,\bullet}X\right) \;, \qquad \overline{\square}_g \;:=\; \left[\overline{\partial},\,\overline{\partial}^*\right] \;\in\; \mathrm{End}^{0,0}\left(\wedge^{\bullet,\bullet}X\right) \;,$$

and the 4th order self-$\langle\cdot\,|\,\cdot\cdot\rangle$-adjoint elliptic differential operators, [46, Proposition 5], [64, §2.b, §2.c],

$$\tilde\Delta^{BC}_g \;:=\; \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \;\in\; \mathrm{End}^{0,0}\left(\wedge^{\bullet,\bullet}X\right)$$

and

$$\tilde\Delta^{A}_g \;:=\; \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* \;\in\; \mathrm{End}^{0,0}\left(\wedge^{\bullet,\bullet}X\right) \;,$$

(from now on, the metric g will be understood whenever it is clear from the context,) induce the $\langle\cdot\,|\,\cdot\cdot\rangle$-orthogonal decompositions, [45, page 450],

$$\wedge^\bullet X\otimes_{\mathbb{R}}\mathbb{C} \;=\; \ker\Delta\oplus\mathrm{im}\,\Delta \;=\; \ker\Delta\oplus\mathrm{im}\,\mathrm{d}\oplus\mathrm{im}\,\mathrm{d}^*$$

and

$$\begin{aligned} \wedge^{\bullet,\bullet}X \;&=\; \ker\square\oplus\mathrm{im}\,\square \;=\; \ker\square\oplus\mathrm{im}\,\partial\oplus\mathrm{im}\,\partial^* \\ &=\; \ker\overline{\square}\oplus\mathrm{im}\,\overline{\square} \;=\; \ker\overline{\square}\oplus\mathrm{im}\,\overline{\partial}\oplus\mathrm{im}\,\overline{\partial}^* \;, \end{aligned}$$

and, [64, Théorème 2.2, §2.c],

$$\begin{aligned}\wedge^{\bullet,\bullet}X &= \ker\tilde{\Delta}^{BC}\oplus\operatorname{im}\tilde{\Delta}^{BC} \;=\; \ker\tilde{\Delta}^{BC}\oplus\operatorname{im}\partial\overline{\partial}\oplus\left(\operatorname{im}\partial^{*}+\operatorname{im}\overline{\partial}^{*}\right)\\ &= \ker\tilde{\Delta}^{A}\oplus\operatorname{im}\tilde{\Delta}^{A} \;=\; \ker\tilde{\Delta}^{A}\oplus\left(\operatorname{im}\partial+\operatorname{im}\overline{\partial}\right)\oplus\operatorname{im}\left(\partial\overline{\partial}\right)^{*}\;.\end{aligned}$$

In particular, by arguing as in Lemma 1.5, it follows that

$$H^{\bullet}_{dR}(X;\mathbb{C}) := \frac{\ker\mathrm{d}}{\operatorname{im}\mathrm{d}}\simeq\ker\Delta\,,\qquad H^{\bullet,\bullet}_{\partial}(X) := \frac{\ker\partial}{\operatorname{im}\partial}\simeq\ker\square\,,\qquad H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\ker\overline{\partial}}{\operatorname{im}\overline{\partial}}\simeq\ker\overline{\square}\,,$$

and, [64, Corollaire 2.3, §2.c],

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}\simeq\ker\tilde{\Delta}^{BC}\,,\qquad H^{\bullet,\bullet}_{A}(X) := \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}}\simeq\ker\tilde{\Delta}^{A}\,.$$

Note that $\bar{*}_g\circ\tilde{\Delta}^{BC}=\tilde{\Delta}^{A}\circ\bar{*}_g$, and hence the Hodge-$*$-operator induces the isomorphism

$$H^{\bullet,\bullet}_{BC}(X)\stackrel{\simeq}{\to}H^{n-\bullet,n-\bullet}_{A}(X)\,.$$

In particular, by Proposition 2.1, one gets straightforwardly the following result, which provides a condition under which the Bott-Chern cohomology of a finite-dimensional sub-complex of $\wedge^{\bullet,\bullet}X$ is a subgroup of $H^{\bullet,\bullet}_{BC}(X)$. Such a result will be applied in the next section with the aim to study the Bott-Chern cohomology of a certain class of solvmanifolds.

Proposition 2.2. *Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g. Let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ be a finite-dimensional sub-complex of $\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ having a structure of bi-differential $\mathbb{Z}^2$-graded algebra of* PD*-type of* PD*-dimension n induced by $\wedge^{\bullet,\bullet}X$. Suppose that*

$$\bar{*}_g\lfloor_{C^{\bullet,\bullet}}\colon C^{\bullet,\bullet}\to C^{n-\bullet,n-\bullet}\,.$$

Then, for any $(p,q)\in\mathbb{Z}^2$, the induced inclusions

$$\left(\mathrm{Tot}^{\bullet}\left(C^{\bullet,\bullet}\right),\,\partial+\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)\,,\qquad\left(C^{\bullet,q},\,\partial\right)\hookrightarrow\left(\wedge^{\bullet,q}X,\,\partial\right)\,,\qquad\left(C^{p,\bullet},\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{p,\bullet}X,\,\overline{\partial}\right)\,,$$

and

$$\left(C^{p-1,q-1}\stackrel{\partial\overline{\partial}}{\to}C^{p,q}\stackrel{\partial+\overline{\partial}}{\to}C^{p+1,q}\oplus C^{p,q+1}\right)\hookrightarrow\left(\wedge^{p-1,q-1}X\stackrel{\partial\overline{\partial}}{\to}\wedge^{p,q}X\stackrel{\partial+\overline{\partial}}{\to}\wedge^{p+1,q}X\oplus\wedge^{p,q+1}X\right)$$

and

$$\left(C^{p-1,q}\oplus C^{p,q-1}\stackrel{(\partial,\overline{\partial})}{\to}C^{p,q}\stackrel{\partial\overline{\partial}}{\to}C^{p+1,q+1}\right)\hookrightarrow\left(\wedge^{p-1,q}X\oplus\wedge^{p,q-1}X\stackrel{(\partial,\overline{\partial})}{\to}\wedge^{p,q}X\stackrel{\partial\overline{\partial}}{\to}\wedge^{p+1,q+1}X\right)$$

induce injective maps in cohomology.

Proof. The proof follows straightforwardly by [64, Théorème 2.2, §2.c] and [45, page 450], and by Proposition 2.1. $\square$

Remark 2.3. By applying Corollary 1.7 to the $\langle\cdot\,|\,\cdot\cdot\rangle$-completion of $\wedge^{\bullet,\bullet}X$, the same conclusion of Proposition 2.2 holds true for a (possibly non-finite-dimensional) closed sub-complex $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ such that $\pi^{C^{\bullet,\bullet}}\circ\partial=\partial\circ\pi^{C^{\bullet,\bullet}}\colon\wedge^{\bullet,\bullet}X\to C^{\bullet,\bullet}$ and $\pi^{C^{\bullet,\bullet}}\circ\overline{\partial}=\overline{\partial}\circ\pi^{C^{\bullet,\bullet}}\colon\wedge^{\bullet,\bullet}X\to C^{\bullet,\bullet}$.

In order to study cohomologies of solvmanifolds, we need also the following result.

To simplify the notation, we say that a sub-complex $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ *suffices in computing the de Rham, respectively conjugate Dolbeault, respectively Dolbeault, respectively Bott-Chern, respectively Aeppli cohomology of X* if the induced inclusion

$$\left(\mathrm{Tot}^{\bullet}C^{\bullet,\bullet},\,\partial+\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)\,,$$

respectively, for any $q\in\mathbb{N}$,

$$\left(C^{\bullet,q},\,\partial\right)\hookrightarrow\left(\wedge^{\bullet,q},\,\partial\right)\,,$$

respectively, for any $p\in\mathbb{N}$,

$$\left(C^{p,\bullet},\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{p,\bullet},\,\overline{\partial}\right)\,,$$

respectively, for any $(p,q)\in\mathbb{Z}^2$,

$$\left(C^{p-1,q-1}\stackrel{\partial\overline{\partial}}{\to}C^{p,q}\stackrel{\partial+\overline{\partial}}{\to}C^{p+1,q}\oplus C^{p,q+1}\right)\hookrightarrow\left(\wedge^{p-1,q-1}X\stackrel{\partial\overline{\partial}}{\to}\wedge^{p,q}X\stackrel{\partial+\overline{\partial}}{\to}\wedge^{p+1,q}X\oplus\wedge^{p,q+1}X\right)$$

respectively, for any $(p,q)\in\mathbb{Z}^2$,

$$\left(C^{p-1,q}\oplus C^{p,q-1}\stackrel{(\partial,\overline{\partial})}{\to}C^{p,q}\stackrel{\partial\overline{\partial}}{\to}C^{p+1,q+1}\right)\hookrightarrow\left(\wedge^{p-1,q}X\oplus\wedge^{p,q-1}X\stackrel{(\partial,\overline{\partial})}{\to}\wedge^{p,q}X\stackrel{\partial\overline{\partial}}{\to}\wedge^{p+1,q+1}X\right)$$

is a quasi-isomorphism.

Proposition 2.4. *Let X be a compact complex manifold of complex dimension n endowed with a Hermitian metric g. Let $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ be a finite-dimensional sub-complex of $\left(\wedge^{\bullet,\bullet}X,\,\partial,\,\overline{\partial}\right)$ having a structure of bi-differential $\mathbb{Z}^2$-graded algebra of* PD*-type of* PD*-dimension n induced by $\wedge^{\bullet,\bullet}X$ and such that*

$$\bar{*}_g\lfloor_{C^{\bullet,\bullet}}\colon C^{\bullet,\bullet}\to C^{n-\bullet,n-\bullet}\;.$$

Let $\left(B^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ be a sub-complex of $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ having a structure of bi-differential $\mathbb{Z}^2$-graded algebra of PD*-type of* PD*-dimension n induced by $C^{\bullet,\bullet}$ and such that*

$$\bar{*}_g\lfloor_{B^{\bullet,\bullet}}\colon B^{\bullet,\bullet}\to B^{n-\bullet,n-\bullet}\;.$$

If $\left(B^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ suffices in computing the cohomologies of X, then also $\left(C^{\bullet,\bullet},\,\partial,\,\overline{\partial}\right)$ suffices in computing the corresponding cohomologies of X.

Proof. By Proposition 2.1 and Proposition 2.2, both the inclusions $B^{\bullet,\bullet}\hookrightarrow C^{\bullet,\bullet}$ and $C^{\bullet,\bullet}\hookrightarrow\wedge^{\bullet,\bullet}X$ induce injective maps in cohomology, whose composition is an isomorphism by the hypothesis. □

2.3. **Complex nilmanifolds.** Let $X=\Gamma\backslash G$ be a *solvmanifold* (respectively, a *nilmanifold*), namely, a compact quotient of a connected simply-connected solvable (respectively, nilpotent) Lie group G by a co-compact discrete subgroup Γ, endowed with a G-left-invariant (almost-)complex structure J. We recall that a solvmanifold is called *completely-solvable* if, for any $g\in G$, all the eigenvalues of $\mathrm{Ad}_g:=\mathrm{d}\left(\psi_g\right)_e\in\mathrm{Aut}(\mathfrak{g})$ are real, equivalently, for any $X\in\mathfrak{g}$, all the eigenvalues of $\mathrm{ad}_X:=[X,\cdot]\in\mathrm{End}(\mathfrak{g})$ are real, where $\psi\colon G\ni g\mapsto\left(\psi_g\colon h\mapsto g\,h\,g^{-1}\right)\in\mathrm{Aut}(G)$ and e is the identity element of G.

Recall that, by J. Milnor's Lemma [52, Lemma 6.2], G is unimodular (that is, $\det(\mathrm{Ad}_g)=1$ for any $g\in G$), and hence, in particular, there exists a G-bi-invariant volume form η on X such that $\int_X\eta=1$. Therefore, consider the *F. A. Belgun symmetrization map* in [14, Theorem 7], namely,

$$\mu\colon\;\wedge^\bullet X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^\bullet\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\;,\qquad\mu(\alpha)\;:=\;\int_X\alpha\lfloor_x\,\eta(x)\;.$$

Note, [14, Theorem 7], that μ commutes with d and with J, and hence also with ∂ and $\overline{\partial}$, and that $\mu\lfloor_{\wedge^\bullet(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^*}=\mathrm{id}_{\wedge^\bullet(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^*}$.

Lemma 2.5. *Let $\Gamma\backslash G$ be a solvmanifold, and consider the F. A. Belgun symmetrization map $\mu\colon\wedge^\bullet X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^\bullet\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*$ in* [14, Theorem 7]*. For a G-left-invariant differential form θ on $\Gamma\backslash G$ and for a differential form ω on $\Gamma\backslash G$, we have*

$$\mu(\theta\wedge\omega)=\theta\wedge\mu(\omega).$$

Proof. Suppose that θ is a G-left-invariant 1-form on $\Gamma\backslash G$. Let ω be a p-form on $\Gamma\backslash G$. Then for $X_1,\ldots,X_{p+1}\in\mathfrak{g}$, since $\theta(X_j)$ is constant for every $j\in\{1,\ldots,p+1\}$, we have

$$\begin{aligned}\mu(\theta\wedge\omega)(X_1,\ldots,X_{p+1})\;&=\;\int_{\Gamma\backslash G}\sum_{\sigma\in\mathfrak{S}_{p+1}}\theta_x\left(X_{\sigma(1)}\right)\cdot\omega\left(X_{\sigma(2)},\ldots,X_{\sigma(p+1)}\right)\,\eta(x)\\&=\;\sum_{\sigma\in\mathfrak{S}_{p+1}}\theta\left(X_{\sigma(1)}\right)\cdot\int_{\Gamma\backslash G}\omega_x\left(X_{\sigma(2)},\ldots,X_{\sigma(p+1)}\right)\,\eta(x)\\&=\;\left(\theta\wedge\mu(\omega)\right)\left(X_1,\ldots,X_{p+1}\right)\;,\end{aligned}$$

where $\mathfrak{S}_{p+1}$ is the set of permutations of $p+1$ elements. Hence, in this case, the lemma holds. We can easily check that the lemma holds in the general case. □

Lemma 2.6 (see [11, Proposition 5.4])**.** *Let $X=\Gamma\backslash G$ be a completely-solvable solvmanifold endowed with a G-left-invariant complex structure J. Consider the sub-complex*

$$j\colon\;\left(\wedge^\bullet\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*,\,\mathrm{d}\right)\hookrightarrow\left(\wedge^\bullet X\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)\;,$$

which is a quasi-isomorphism by A. Hattori's theorem [37, Corollary 4.2]*. The induced map*

$$
\begin{aligned}
j\colon & \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)\cap\wedge^{p,q}\mathfrak{g}^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)} \\
& \to \quad \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)}
\end{aligned}
$$

is an isomorphism.

Proof. For the sake of completeness, we recall here the argument of the proof (note that the statement holds, more in general, in the almost-complex setting).

The F. A. Belgun symmetrization map $\mu\colon \wedge^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}$ induces the map

$$
\begin{aligned}
\mu\colon & \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \\
& \to \quad \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)\cap\wedge^{p,q}\mathfrak{g}^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q}\left(\mathfrak{g}\right)^{*}\otimes_{\mathbb{R}}\mathbb{C}\right)}\ .
\end{aligned}
$$

Hence, one gets the commutative diagram

$$
\begin{tikzcd}
\frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\to\wedge^{p+q+1}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\right)\cap\wedge^{p,q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\to\wedge^{p+q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\right)} \arrow[d, "j"] \arrow[dd, bend right=60, "\mathrm{id}"'] \\
\frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \arrow[d, "\mu"] \\
\frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\to\wedge^{p+q+1}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\right)\cap\wedge^{p,q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\to\wedge^{p+q}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^{*}\right)}
\end{tikzcd}
,
$$

from which one gets that

$$
\begin{aligned}
j\colon & \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)\cap\wedge^{p,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)} \\
& \to \quad \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)}
\end{aligned}
$$

is injective, and that

$$
\begin{aligned}
\mu\colon & \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \\
& \to \quad \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)\cap\wedge^{p,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)}
\end{aligned}
$$

is surjective.

Moreover, since $j\colon \left(\wedge^{\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*},\, \mathrm{d}\right)\hookrightarrow\left(\wedge^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C},\, \mathrm{d}\right)$ is a quasi-isomorphism by A. Hattori's theorem [37, Theorem 4.2], one gets that $\mu\colon H^{\bullet}_{dR}(X;\mathbb{C})\to H^{\bullet}\left(\wedge^{\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*},\, \mathrm{d}\right)$ is in fact the identity map, and hence

$$
\begin{aligned}
\mu\colon & \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}X}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \\
& \to \quad \frac{\ker\left(\mathrm{d}\colon \wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)\cap\wedge^{p,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}}{\operatorname{im}\left(\mathrm{d}\colon \wedge^{p+q-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\to\wedge^{p+q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^{*}\right)}
\end{aligned}
$$

is also injective.

Since X is compact, the dimension of $H^{\bullet}_{dR}(X;\mathbb{C})$ is finite, and hence μ is in fact an isomorphism. □

As an application of Theorem 1.3 and Proposition 2.2, one recovers the following results, concerning the Bott-Chern cohomology of nilmanifolds. (We refer to [71, 54, 13, 3, 25, 22, 59, 62] for definitions and notation.)

Corollary 2.7 ([4, Theorem 3.8]). *Let $X = \Gamma\backslash G$ be a nilmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that one of the following conditions holds:*

- *X is complex parallelizable;*
- *J is an Abelian complex structure;*
- *J is a nilpotent complex structure;*
- *J is a rational complex structure;*
- *$\mathfrak{g}$ admits a torus-bundle series compatible with J and with the rational structure induced by Γ;*
- *$\dim_{\mathbb{R}} \mathfrak{g} = 6$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{h}_7 := \left(0^3,\, 12,\, 13,\, 23\right)$.*

Then the inclusion $j\colon \left(\wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*,\, \partial,\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ induces the isomorphisms

$$H^{\bullet,\bullet}_{BC}(X) \;\simeq\; \frac{\ker\left(\mathrm{d}\colon\, \wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \to \wedge^{\bullet+\bullet+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon\, \wedge^{\bullet-1,\bullet-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \to \wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right)}$$

and

$$H^{\bullet,\bullet}_{A}(X) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon\, \wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \to \wedge^{\bullet+1,\bullet+1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right)}{\mathrm{im}\left(\partial\colon\, \wedge^{\bullet-1,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \to \wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right) + \mathrm{im}\left(\overline{\partial}\colon\, \wedge^{\bullet,\bullet-1}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \to \wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right)}\;.$$

Proof. Choose a G-left-invariant Hermitian metric g on X. The sub-complex $\left(\wedge^{\bullet,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*,\, \partial,\, \overline{\partial}\right)$ being finite-dimensional, the induced maps in Bott-Chern, respectively Aeppli cohomologies are injective by Proposition 2.2.

Under the hypothesis, by [61, Theorem 1], [25, Main Theorem], [22, Theorem 2, Remark 4], [59, Theorem 1.10], and [60, Corollary 3.10], one has that, for any fixed $p \in \mathbb{N}$, the induced map

$$j\colon\, \left(\wedge^{p,\bullet}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*,\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{p,\bullet}X,\, \overline{\partial}\right)$$

is a quasi-isomorphism. By conjugation, one has also that, for any fixed $q \in \mathbb{N}$, the induced map

$$j\colon\, \left(\wedge^{\bullet,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*,\, \partial\right) \hookrightarrow \left(\wedge^{\bullet,q}X,\, \partial\right)$$

is a quasi-isomorphism. Lastly, condition *(iii)* in Theorem 1.3 is satisfied by Lemma 2.6. Hence, by Theorem 1.3, the induced map in Bott-Chern cohomology is surjective.

As regards Aeppli cohomologies, it suffices to note that the Hodge-$*$-operator $\bar{*}_g$ induces the isomorphisms $H^{\bullet,\bullet}_{BC}(X) \stackrel{\simeq}{\to} H^{n-\bullet,n-\bullet}_{A}(X)$ and $\frac{\ker \mathrm{d}\lfloor_{\wedge^{\bullet,\bullet}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^*}}{\mathrm{im}\,\partial\overline{\partial}} \stackrel{\simeq}{\to} \frac{\ker \partial\overline{\partial}\lfloor_{\wedge^{n-\bullet,n-\bullet}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})^*}}{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}}$, where n is the complex dimension of X. □

The previous result can be used to compute the cohomology of the left-invariant complex structures classified by M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa in [20], as in [6] and [48].

2.4. **Complex solvmanifolds.** Let G be a connected simply-connected n-dimensional solvable Lie group admitting a discrete co-compact subgroup Γ, and denote by $\mathfrak{g}$ the (solvable) Lie algebra of G. Set $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$.

Consider the *adjoint action*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\;, \qquad \mathrm{ad}_X \;:=\; [X, \cdot]\;;$$

by denoting by $\mathrm{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \;:\; \forall X \in \mathfrak{g},\; [D, \mathrm{ad}_X] = \mathrm{ad}_{DX}\}$ the $\mathbb{R}$-vector space of *derivations* on $\mathfrak{g}$, one has that $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$. One has that every derivation ad_X, for $X \in \mathfrak{g}$, admits a unique *Jordan decomposition*, see, e.g., [32, II.1.10], namely,

$$\mathrm{ad}_X \;=\; \left(\mathrm{ad}_X\right)_{\mathrm{s}} + \left(\mathrm{ad}_X\right)_{\mathrm{n}}\;,$$

where $\left(\mathrm{ad}_X\right)_{\mathrm{s}} \in \mathfrak{gl}(\mathfrak{g})$ is *semi-simple* (that is, each $\left(\mathrm{ad}_X\right)_{\mathrm{s}}$-invariant sub-space of $\mathfrak{g}$ admits an $\left(\mathrm{ad}_X\right)_{\mathrm{s}}$-invariant complementary sub-space in $\mathfrak{g}$), and $\left(\mathrm{ad}_X\right)_{\mathrm{n}} \in \mathfrak{gl}(\mathfrak{g})$ is *nilpotent* (that is, there exists $N \in \mathbb{N}$ such that $\left(\mathrm{ad}_X\right)_{\mathrm{n}}^N = 0$).

Let $\mathfrak{n}$ be the *nilradical* of $\mathfrak{g}$, that is, the maximal nilpotent ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, there exists an $\mathbb{R}$-vector sub-space V (which is not necessarily a Lie algebra) of $\mathfrak{g}$ so that *(i)* $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of $\mathbb{R}$-vector spaces, and, *(ii)* for any $A, B \in V$, it holds that $\left(\mathrm{ad}_A\right)_{\mathrm{s}}(B) = 0$, see, e.g., [32, Proposition II I.1.1]. Hence, one can define the map

$$\mathrm{ad}_{\mathrm{s}}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})\;, \qquad \mathfrak{g} = V \oplus \mathfrak{n} \ni (A, X) \mapsto \left(\mathrm{ad}_{\mathrm{s}}\right)_{A+X} := \left(\mathrm{ad}_A\right)_{\mathrm{s}} \in \mathrm{Der}(\mathfrak{g})\;.$$

Moreover, one has that *(iii)* $[\mathrm{ad}_{\mathrm{s}}(\mathfrak{g}), \mathrm{ad}_{\mathrm{s}}(\mathfrak{g})] = \{0\}$, and *(iv)* $\mathrm{ad}_{\mathrm{s}}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is $\mathbb{R}$-linear, see, e.g., [32, Proposition III.1.1].

Since we have $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{n}$, see, e.g., [32, II.1.9], and $\mathrm{ad_s}(\mathfrak{n}) = \{0\}$, the map $\mathrm{ad_s}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of $\mathfrak{g}$, whose image $\mathrm{ad_s}(\mathfrak{g})$ is Abelian and consists of semi-simple elements. Hence, denote by

$$\mathrm{Ad_s}\colon G \to \mathrm{Aut}(\mathfrak{g}) \;, \qquad \text{respectively } \mathrm{Ad_s}\colon G \to \mathrm{Aut}\left(\mathfrak{g}_{\mathbb{C}}\right) \;,$$

the unique representation which lifts $\mathrm{ad_s}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, see, e.g., [72, Theorem 3.27], respectively the natural $\mathbb{C}$-linear extension.

Let T be the Zariski-closure of $\mathrm{Ad_s}(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. Denote by $\mathrm{Char}(T) := \mathrm{Hom}(T;\mathbb{C}^*)$ the set of all 1-dimensional algebraic group representations of T. Set

$$\mathcal{C}_\Gamma \;:=\; \left\{\beta\circ\mathrm{Ad_s} \in \mathrm{Hom}\left(G;\mathbb{C}^*\right) \;:\; \beta\in\mathrm{Char}(T),\; \left(\beta\circ\mathrm{Ad_s}\right)\lfloor_\Gamma = 1\right\} \;.$$

We consider the differential graded sub-algebra

$$\bigoplus_{\alpha\in\mathcal{C}_\Gamma} \alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$$

of $\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}$. (Note that we have used left-translations on G to identify the elements of $\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$ with the G-left-invariant complex forms in $\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}$, namely, the complex forms being invariant for the action of the Lie group G on $\Gamma\backslash G$ given by left-translations.) By $\mathrm{Ad_s}(G)\subseteq\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ we have the $\mathrm{Ad_s}(G)$-action on the differential graded algebra $\bigoplus_{\alpha\in\mathcal{C}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$. We denote by $A_\Gamma^\bullet$ the space consisting of the $\mathrm{Ad_s}(G)$-invariant elements of $\bigoplus_{\alpha\in\mathcal{C}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$, namely,

$$A_\Gamma^\bullet \;:=\; \left\{\varphi\in\bigoplus_{\alpha\in\mathcal{C}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^* \;:\; \left(\mathrm{Ad_s}\right)_g(\varphi)=\varphi \text{ for every } g\in G\right\} \;. \tag{1}$$

Now we consider the inclusion

$$A_\Gamma^\bullet \subseteq \wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}$$

of differential graded algebras. We have the following result.

Theorem 2.8 ([39, Corollary 7.6])**.** *Let $\Gamma\backslash G$ be a solvmanifold, and consider $A_\Gamma^\bullet$ as defined in* (1)*. Then the inclusion*

$$\left(A_\Gamma^\bullet,\,\mathrm{d}\right) \hookrightarrow \left(\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)$$

of differential graded algebras induces an isomorphism in cohomology.

Note that $\mathrm{Ad_s}(G)\subseteq\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ consists of simultaneously diagonalizable elements. Let $\{X_1,\ldots,X_n\}$ be a basis of $\mathfrak{g}_{\mathbb{C}}$ with respect to which

$$\mathrm{Ad_s} \;=\; \mathrm{diag}\left(\alpha_1,\ldots,\alpha_n\right) : G\to\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$$

for some characters

$$\alpha_1\in\mathrm{Hom}(G;\mathbb{C}^*),\ldots,\alpha_n\in\mathrm{Hom}(G;\mathbb{C}^*) \;.$$

Let $\{x_1,\ldots,x_n\}$ be the dual basis of $\mathfrak{g}_{\mathbb{C}}^*$ of $\{X_1,\ldots,X_n\}$. For the basis $\left\{x_{i_1}\wedge\cdots\wedge x_{i_p}\right\}_{1\leq i_1<i_2<\cdots<i_p\leq n}$ of $\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$, for $\alpha\in\mathcal{C}_\Gamma$, we have

$$\left(\mathrm{Ad_s}\right)_g\left(\alpha\,x_{i_1}\wedge\cdots\wedge x_{i_p}\right) \;=\; \alpha(g)\,\alpha_{i_1\cdots i_p}^{-1}(g)\,\alpha\,x_{i_1}\wedge\cdots\wedge x_{i_p} \;,$$

where we have shortened $\alpha_{i_1\cdots i_p} := \alpha_{i_1}\cdot\cdots\cdot\alpha_{i_p}\in\mathrm{Hom}\left(G;\mathbb{C}^*\right)$. Then the basis

$$\left\{\alpha\,x_{i_1}\wedge\cdots\wedge x_{i_p} \;\middle|\; 1\leq i_1<i_2<\cdots<i_p\leq n \text{ and } \alpha\in\mathcal{C}_\Gamma\right\}$$

of $\bigoplus_{\alpha\in\mathcal{C}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*$ diagonalizes the $\mathrm{Ad_s}(G)$-action, and $\alpha\,x_{i_1}\wedge\cdots\wedge x_{i_p}\in A_\Gamma^\bullet$ if and only if $\alpha=\alpha_{i_1\cdots i_p}$ and $\alpha_{i_1\cdots i_p}\lfloor_\Gamma = 1$. Hence the differential graded algebra $A_\Gamma^\bullet$ is written as

$$A_\Gamma^p \;=\; \mathbb{C}\left\langle \alpha_{i_1\cdots i_p}\,x_{i_1}\wedge\cdots\wedge x_{i_p} \;\middle|\; 1\leq i_1<i_2<\cdots<i_p\leq n \;\text{ such that } \alpha_{i_1\cdots i_p}\lfloor_\Gamma = 1\right\rangle \;. \tag{2}$$

In fact, the following result holds.

Theorem 2.9. *Let $\Gamma\backslash G$ be a solvmanifold. Let $\{X_1,\ldots,X_n\}$ be a basis of the $\mathbb{C}$-vector space $\mathfrak{g}_{\mathbb{C}}$ with respect to which $\mathrm{Ad_s}=\mathrm{diag}\left(\alpha_1,\ldots,\alpha_n\right)$ for some characters $\alpha_1,\ldots,\alpha_n\in\mathrm{Hom}(G;\mathbb{C}^*)$. Consider the finite set of characters*

$$\mathcal{A}_\Gamma \;:=\; \left\{\alpha_{i_1\cdots i_p}\in\mathrm{Hom}(G;\mathbb{C}^*) \;:\; 1\leq i_1<i_2<\cdots<i_p\leq n \;\text{ such that } \alpha_{i_1\cdots i_p}\lfloor_\Gamma = 1\right\} \;.$$

Then the sub-complex

$$\iota\colon \left(\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}_{\mathbb{C}}^*,\,\mathrm{d}\right) \hookrightarrow \left(\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)$$

induces an isomorphism in cohomology.

Suppose furthermore that G is endowed with a G-left-invariant complex structure. Consider the bi-graded $\mathbb{C}$-vector sub-space

$$\iota\colon \bigoplus_{\alpha\in\mathcal{A}_\Gamma} \alpha\cdot\wedge^{\bullet,\bullet}\mathfrak{g}^*_{\mathbb{C}} \hookrightarrow \wedge^{\bullet,\bullet}\,\Gamma\backslash G \;;$$

then ι induces, for any $(p,q)\in\mathbb{Z}^2$, the isomorphism

$$\iota^*\colon \frac{\ker \mathrm{d}\lfloor_{\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}}}}{\mathrm{d}\left(\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^{p+q-1}\mathfrak{g}^*_{\mathbb{C}}\right)} \xrightarrow{\simeq} \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}\right)} \;.$$

Proof. Consider the G-left-invariant Hermitian metric

$$g \;:=\; \sum_{j=1}^{n} x_j\odot\bar{x}_j$$

on $\Gamma\backslash G$, and the associated $\mathbb{C}$-anti-linear Hodge-$*$-operator $\bar{*}_g\colon \wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^{n-\bullet}\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}$, where n is the dimension of $\Gamma\backslash G$. If the restriction of a character α of G on Γ is trivial, then α induces a function on $\Gamma\backslash G$ and the image $\alpha(G)$ is a compact subgroup of $\mathbb{C}^*$, and hence α is unitary. For $\alpha_{i_1\cdots i_p}:=\alpha_{i_1}\cdot\dots\cdot\alpha_{i_p}\in\mathcal{A}_\Gamma$, since G is unimodular, [52, Lemma 6.2], for the complement $\{j_1,\dots,j_{n-p}\}:=\{1,\dots,n\}\setminus\{i_1,\dots,i_p\}$ we have

$$\bar{\alpha}_{i_1\dots i_p}=\alpha^{-1}_{i_1\cdots i_p}=\alpha_{j_1\dots j_{n-p}}.$$

By this, we have

$$\bar{*}_g\left(\alpha_{i_1\cdots i_p}\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\right) \;=\; \alpha_{j_1\dots j_{n-p}}\cdot\wedge^{n-\bullet}\mathfrak{g}^*_{\mathbb{C}}$$

and, for $\alpha_{i_1\dots i_p}\,x_{i_1}\wedge\dots\wedge x_{i_p}\in A^\bullet_\Gamma$, we have

$$\bar{*}_g\left(\alpha_{i_1\dots i_p}\,x_{i_1}\wedge\dots\wedge x_{i_p}\right) \;=\; \alpha_{j_1\dots j_{n-p}}\,x_{j_1}\wedge\dots\wedge x_{j_{n-p}}\in A^{n-\bullet}_\Gamma \;.$$

Hence the sub-complexes

$$\left(A^\bullet_\Gamma,\,\mathrm{d}\right)\hookrightarrow\left(\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}},\,\mathrm{d}\right)\hookrightarrow\left(\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C},\,\mathrm{d}\right)$$

are such that

$$\bar{*}_g\lfloor_{A^\bullet_\Gamma}\colon A^\bullet_\Gamma\to A^{n-\bullet}_\Gamma \qquad\text{and}\qquad \bar{*}_g\lfloor_{\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}}\colon \bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\to\bigoplus_{\alpha\in\mathcal{A}_\Gamma}\alpha\cdot\wedge^{n-\bullet}\mathfrak{g}^*_{\mathbb{C}} \;,$$

therefore the first assertion follows from Theorem 2.8 and Proposition 2.4.

Consider the F. A. Belgun symmetrization map $\mu\colon \wedge^\bullet X\otimes_{\mathbb{R}}\mathbb{C}\to\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}$, [14, Theorem 7]. For $\alpha\in\mathcal{A}_\Gamma$, we define the map

$$\varphi_\alpha\colon \wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}\to\alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\;,\qquad \varphi_\alpha(\omega)\;:=\;\alpha\cdot\mu\left(\frac{\omega}{\alpha}\right)\;.$$

By the definition of μ, for a G-left-invariant differential form θ on $\Gamma\backslash G$ and for a differential form ω on $\Gamma\backslash G$, we have $\mu(\theta\wedge\omega)=\theta\wedge\mu(\omega)$, see Lemma 2.5. By this we have, for any $\alpha\in\mathcal{A}_\Gamma$,

$$\begin{aligned}\varphi_\alpha(\mathrm{d}\,\omega) &= \alpha\cdot\mu\left(\frac{\mathrm{d}\,\omega}{\alpha}\right) \;=\; \alpha\cdot\mu\left(\mathrm{d}\left(\frac{\omega}{\alpha}\right)+\frac{\mathrm{d}\,\alpha}{\alpha}\wedge\frac{\omega}{\alpha}\right)\\ &= \alpha\cdot\mathrm{d}\,\mu\left(\frac{\omega}{\alpha}\right)+\mathrm{d}\,\alpha\wedge\mu\left(\frac{\omega}{\alpha}\right) \;=\; \mathrm{d}\left(\alpha\cdot\mu\left(\frac{\omega}{\alpha}\right)\right)\\ &= \mathrm{d}\,\varphi_\alpha(\omega)\;,\end{aligned}$$

and hence φ_α is a morphism of cochain complexes. Furthermore, for $\alpha\in\mathcal{A}_\Gamma$, by considering the inclusion

$$\iota_\alpha\colon \alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}\hookrightarrow\wedge^\bullet\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}\;,$$

we have that

$$\varphi_\alpha\circ\iota_\alpha \;=\; \mathrm{id}_{\alpha\cdot\wedge^\bullet\mathfrak{g}^*_{\mathbb{C}}}\;.$$

For distinct characters $\alpha,\alpha'\in\mathcal{A}_\Gamma$, for the G-left-invariant form $\frac{\alpha'}{\alpha}\,\mathrm{d}\left(\frac{\alpha}{\alpha'}\right)$, since η is a G-left-invariant volume form, we can choose $\lambda\in\wedge^{\dim G-1}\mathfrak{g}^*_{\mathbb{C}}$ such that $\frac{\alpha'}{\alpha}\,\mathrm{d}\left(\frac{\alpha}{\alpha'}\right)\wedge\lambda=\eta$. Then we have

$$\mathrm{d}\left(\frac{\alpha}{\alpha'}\lambda\right) \;=\; \frac{\alpha}{\alpha'}\,\frac{\alpha'}{\alpha}\,\mathrm{d}\left(\frac{\alpha}{\alpha'}\right)\wedge\lambda \;=\; \frac{\alpha}{\alpha'}\,\eta\;.$$

By this, using Stokes' theorem, for $\alpha\,\omega \in \alpha \cdot \wedge^p \mathfrak{g}_{\mathbb{C}}^*$ and for $X_1, \ldots, X_p \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, we have

$$\begin{aligned}\mu\left(\frac{\alpha}{\alpha'}\,\omega\right)(X_1, \ldots, X_p) &= \int_{\Gamma\backslash G} \frac{\alpha(x)}{\alpha'(x)}\,\omega\lfloor_x (X_1\lfloor_x, \ldots, X_p\lfloor_x)\,\eta(x) \;=\; \omega\,(X_1, \ldots X_p) \int_{\Gamma\backslash G} \frac{\alpha(x)}{\alpha'(x)}\,\eta(x) \\ &= \omega\,(X_1, \ldots X_p) \int_{\Gamma\backslash G} \mathrm{d}\left(\frac{\alpha}{\alpha'}\,\lambda\right) \;=\; 0\end{aligned}$$

and hence we have

$$\varphi_{\alpha'} \circ \iota_\alpha \;=\; 0\,.$$

By the definition and since the complex structure on $\Gamma\backslash G$ is G-left-invariant, we have that, for any $\alpha \in \mathcal{A}_\Gamma$, for any $(p,q) \in \mathbb{Z}^2$,

$$\varphi_\alpha\left(\wedge^{p,q}\,\Gamma\backslash G\right) \subseteq \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^*\,.$$

By noting that the set $\mathcal{A}_\Gamma$ is finite, we define the map

$$\Phi \;:=\; \sum_{\alpha \in \mathcal{A}_\Gamma} \varphi_\alpha \colon \wedge^{\bullet,\bullet}\,\Gamma\backslash G \to \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*\,;$$

note that Φ is a morphism of cochain complexes and we have, for any $(p,q) \in \mathbb{Z}^2$,

$$\Phi\left(\wedge^{p,q}\,\Gamma\backslash G\right) \subseteq \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^* \qquad \text{and} \qquad \Phi \circ \iota \;=\; \mathrm{id}_{\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^*}\,,$$

where ι denotes the inclusion $\iota\colon \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^* \hookrightarrow \wedge^{\bullet,\bullet}\,\Gamma\backslash G$. Consider the induced maps

$$\iota^*\colon H^\bullet\left(\mathrm{Tot}^\bullet \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*,\, \mathrm{d}\right) \to H^\bullet_{dR}\left(\Gamma\backslash G\,;\mathbb{C}\right)$$

and

$$\Phi^*\colon H^\bullet_{dR}\left(\Gamma\backslash G\,;\mathbb{C}\right) \to H^\bullet\left(\mathrm{Tot}^\bullet \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*,\, \mathrm{d}\right)\,.$$

Since ι^* is an isomorphism by the first assertion and $\Phi^* \circ \iota^* = \mathrm{id}$, then Φ^* is the inverse of ι^*. By $\Phi\left(\wedge^{p,q}\,\Gamma\backslash G\right) \subseteq \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^*$, we have

$$\Phi^*\left(\frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\,\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\,\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}\right)}\right) \;\subseteq\; \frac{\ker \mathrm{d}\lfloor_{\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^*}}{\mathrm{d}\left(\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p+q-1} \mathfrak{g}_{\mathbb{C}}^*\right)}\,.$$

Hence the restriction of Φ^* to $\frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\,\Gamma\backslash G}}{\mathrm{d}(\wedge^{p+q-1}\,\Gamma\backslash G)}$ is the inverse of the restriction of ι^* to $\frac{\ker \mathrm{d}\lfloor_{\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^*}}{\mathrm{d}\left(\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p+q-1} \mathfrak{g}_{\mathbb{C}}^*\right)}$, which is hence an isomorphism. Therefore the second assertion follows. □

Corollary 2.10. *Let $\Gamma\backslash G$ be a solvmanifold. Let J be a G-left-invariant complex structure on G satisfying, for all $g \in G$,*

$$J \circ \left(\mathrm{Ad_s}\right)_g \;=\; \left(\mathrm{Ad_s}\right)_g \circ J\,.$$

Then, by setting $A_\Gamma^{p,q} := A_\Gamma^\bullet \cap \wedge^{p,q}\,\Gamma\backslash G$ for any $(p,q) \in \mathbb{Z}^2$, we have that the differential graded sub-algebra $(A_\Gamma^\bullet,\, \mathrm{d}) \hookrightarrow (\wedge^\bullet\,\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C},\, \mathrm{d})$ defined in (1) *is actually $\mathbb{Z}^2$-graded,*

$$A_\Gamma^\bullet \;=\; \bigoplus_{p+q=\bullet} A_\Gamma^{p,q}\,,$$

and the inclusion $A_\Gamma^{\bullet,\bullet} \subset \wedge^{p,q}\,\Gamma\backslash G$ induces the isomorphism

$$\frac{\ker \mathrm{d}\lfloor_{A_\Gamma^{p,q}}}{\mathrm{d}\left(A_\Gamma^{p+q-1}\right)} \stackrel{\simeq}{\to} \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\,\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\,\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}\right)}\,.$$

Proof. Consider the $\mathrm{Ad_s}(G)$-action on $\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*$. Then $A_\Gamma^{\bullet,\bullet}$ is the sub-complex that consists of the elements of $\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*$ fixed by this action. Since $\mathrm{Ad_s}$ is diagonalizable, we have the decomposition

$$\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^\bullet \mathfrak{g}_{\mathbb{C}}^* \;=\; A_\Gamma^\bullet \oplus D^\bullet$$

such that $D^\bullet$ is a sub-complex and this decomposition is a direct sum of cochain complexes. By the assumption $J \circ (\mathrm{Ad_s})_g = (\mathrm{Ad_s})_g \circ J$ for any $g \in G$, the $\mathrm{Ad_s}(G)$-action is compatible with the bi-grading $\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}^*_{\mathbb{C}}$. Hence we have in fact

$$\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}^*_{\mathbb{C}} \;=\; A^{\bullet,\bullet}_\Gamma \oplus D^{\bullet,\bullet} \;.$$

Consider the projection $p\colon \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}^*_{\mathbb{C}} \to A^{\bullet,\bullet}_\Gamma$ and the inclusion $\iota\colon A^{\bullet,\bullet}_\Gamma \hookrightarrow \bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{\bullet,\bullet} \mathfrak{g}^*_{\mathbb{C}}$. Then we have $p \circ \iota = \mathrm{id}_{A^{\bullet,\bullet}_\Gamma}$. As similar to the proof of Theorem 2.9, we have that ι induces, for any $(p,q) \in \mathbb{Z}^2$, the isomorphism

$$\iota^* \colon \frac{\ker \mathrm{d}\lfloor_{A^{p,q}_\Gamma}}{\mathrm{d}\left(A^{p+q-1}_\Gamma\right)} \stackrel{\simeq}{\to} \frac{\ker \mathrm{d}\lfloor_{\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p,q} \mathfrak{g}^*_{\mathbb{C}}}}{\mathrm{d}\left(\bigoplus_{\alpha \in \mathcal{A}_\Gamma} \alpha \cdot \wedge^{p+q-1} \mathfrak{g}^*_{\mathbb{C}}\right)} \;.$$

Hence the corollary follows from Theorem 2.9. □

2.4.1. *Complex solvmanifolds of splitting type.* We consider now solvmanifolds of the following type.

Assumption 2.11. *Consider a solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant complex structure J. Assume that G is the semi-direct product $\mathbb{C}^n \ltimes_\phi N$ so that:*

- *(i) N is a connected simply-connected $2m$-dimensional nilpotent Lie group endowed with an N-left-invariant complex structure J_N; (denote the Lie algebras of $\mathbb{C}^n$ and N by $\mathfrak{a}$ and, respectively, $\mathfrak{n}$;)*
- *(ii) for any $t \in \mathbb{C}^n$, it holds that $\phi(t) \in \mathrm{GL}(N)$ is a holomorphic automorphism of N with respect to J_N;*
- *(iii) ϕ induces a semi-simple action on $\mathfrak{n}$;*
- *(iv) G has a lattice Γ; (then Γ can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes_\phi \Gamma_N$ such that $\Gamma_{\mathbb{C}^n}$ and Γ_N are lattices of $\mathbb{C}^n$ and, respectively, N, and, for any $t \in \Gamma'$, it holds $\phi(t)\left(\Gamma_N\right) \subseteq \Gamma_N$;)*
- *(v) the inclusion $\wedge^{\bullet,\bullet}\left(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}\right)^* \hookrightarrow \wedge^{\bullet,\bullet}\left(\Gamma_N\backslash N\right)$ induces the isomorphism*

$$H^\bullet\left(\wedge^{\bullet,\bullet}\left(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}\right)^*, \overline{\partial}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{\overline{\partial}}\left(\Gamma_N\backslash N\right) \;.$$

Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider the decomposition $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$ induced by J_N. By the condition *(ii)*, this decomposition is a direct sum of $\mathbb{C}^n$-modules. By the condition *(iii)*, we have a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^{1,0}$ and characters $\alpha_1, \ldots, \alpha_m \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by

$$\mathbb{C}^n \ni t \mapsto \phi(t) \;=\; \mathrm{diag}\left(\alpha_1(t), \ldots, \alpha_m(t)\right) \in \mathrm{GL}(\mathfrak{n}^{1,0}) \;.$$

For any $j \in \{1, \ldots, m\}$, since Y_j is an N-left-invariant $(1,0)$-vector field on N, the $(1,0)$-vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes_\phi N$ is $(\mathbb{C}^n \ltimes_\phi N)$-left-invariant. Consider the Lie algebra $\mathfrak{g}$ of G and the decomposition $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ induced by J. Hence we have a basis $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$ of $\mathfrak{g}^{1,0}$, and let $\left\{x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m\right\}$ be its dual basis of $\wedge^{1,0}\mathfrak{g}^*_{\mathbb{C}}$. Then we have

$$\wedge^{p,q}\mathfrak{g}^*_{\mathbb{C}} \;=\; \wedge^p \left\langle x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m \right\rangle \otimes \wedge^q \left\langle \bar{x}_1, \ldots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \ldots, \bar{\alpha}_m^{-1} \bar{y}_m \right\rangle \;.$$

The following lemma holds.

Lemma 2.12 ([40, Lemma 2.2])**.** *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J as in Assumption 2.11. Consider a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^{1,0}$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi(t) = \mathrm{diag}\left(\alpha_1(t), \ldots, \alpha_m(t)\right)$ for $\alpha_1, \ldots, \alpha_m \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ characters of $\mathbb{C}^n$. For any $j \in \{1, \ldots, m\}$, there exist unique unitary characters $\beta_j \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ and $\gamma_j \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ on $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \gamma_j^{-1}$ are holomorphic.*

We recall the following result by the second author.

Theorem 2.13. ([40, Corollary 4.2]) *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J as in Assumption 2.11. Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^{1,0}$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi(t) = \mathrm{diag}\left(\alpha_1(t), \ldots, \alpha_m(t)\right)$ for $\alpha_1, \ldots, \alpha_m \in \mathrm{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ characters of $\mathbb{C}^n$. Let $\left\{x_1, \ldots, x_n, \alpha_1^{-1} y_1, \ldots, \alpha_m^{-1} y_m\right\}$ be the basis of $\wedge^{1,0}\mathfrak{g}^*_{\mathbb{C}}$ which is dual to $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$. For any $j \in \{1, \ldots, m\}$, let β_j and γ_j be the unique unitary characters on $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \gamma_j^{-1}$*

are holomorphic, as in Lemma 2.12. Define the differential bi-graded sub-algebra $B_\Gamma^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet}\Gamma\backslash G$, for $(p,q) \in \mathbb{Z}^2$, as

$$B_\Gamma^{p,q} \;:=\; \mathbb{C}\left\langle x_I \wedge \left(\alpha_J^{-1}\beta_J\right) y_J \wedge \bar{x}_K \wedge \left(\bar{\alpha}_L^{-1}\gamma_L\right) \bar{y}_L \;\middle|\; |I|+|J| = p \text{ and } |K|+|L| = q \right. \\ \left. \text{such that } \left(\beta_J\gamma_L\right)\lfloor_\Gamma = 1 \right\rangle \;. \tag{3}$$

Then the inclusion $B_\Gamma^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the cohomology isomorphism

$$H^{\bullet,\bullet}\left(B_\Gamma^{\bullet,\bullet},\, \overline{\partial}\right) \stackrel{\simeq}{\to} H_{\overline{\partial}}^{\bullet,\bullet}(\Gamma\backslash G) \;.$$

As a straightforward consequence, by means of conjugation, we get the following result.

Corollary 2.14. *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J as in Assumption 2.11. Consider $B_\Gamma^{\bullet,\bullet}$ as in* (3)*, and let*

$$\bar{B}_\Gamma^{\bullet,\bullet} \;:=\; \left\{\bar{\omega} \in \wedge^{\bullet,\bullet}\Gamma\backslash G \;:\; \omega \in B_\Gamma^{\bullet,\bullet}\right\} \;. \tag{4}$$

The inclusion $\bar{B}_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the cohomology isomorphism

$$H^{\bullet,\bullet}\left(\bar{B}_\Gamma^{\bullet,\bullet},\, \partial\right) \stackrel{\simeq}{\to} H_{\partial}^{\bullet,\bullet}(\Gamma\backslash G) \;.$$

Hence we get the following result.

Corollary 2.15. *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J as in Assumption 2.11. Consider $B_\Gamma^{\bullet,\bullet}$ as in* (3)*, and $\bar{B}_\Gamma^{\bullet,\bullet}$ as in* (4)*. Let*

$$C_\Gamma^{\bullet,\bullet} \;:=\; B_\Gamma^{\bullet,\bullet} + \bar{B}_\Gamma^{\bullet,\bullet} \;. \tag{5}$$

Then we have

(i) the inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the cohomology isomorphism

$$H^{\bullet,\bullet}\left(C_\Gamma^{\bullet,\bullet},\, \partial\right) \stackrel{\simeq}{\to} H_{\partial}^{\bullet,\bullet}(\Gamma\backslash G) \;;$$

(ii) the inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the cohomology isomorphism

$$H^{\bullet,\bullet}\left(C_\Gamma^{\bullet,\bullet},\, \overline{\partial}\right) \stackrel{\simeq}{\to} H_{\overline{\partial}}^{\bullet,\bullet}(\Gamma\backslash G) \;;$$

(iii) for any $(p,q) \in \mathbb{Z}^2$, the inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the surjective map

$$\frac{\ker \mathrm{d}\lfloor_{C_\Gamma^{p,q}}}{\mathrm{d}\left(\mathrm{Tot}^{p+q-1} C_\Gamma^{\bullet,\bullet}\right)} \to \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}\right)} \;.$$

Proof. Let g be the G-left-invariant Hermitian metric on G defined by

$$g \;:=\; \sum_{j=1}^{n} x_j \odot \bar{x}_j + \sum_{k=1}^{m} \alpha_k^{-1}\bar{\alpha}_k^{-1} y_k \odot \bar{y}_k \;,$$

and consider its associated $\mathbb{C}$-anti-linear Hodge-$*$-operator $\bar{*}_g \colon \wedge^\bullet\Gamma\backslash G \to \wedge^{2N-\bullet}\Gamma\backslash G$, where $2N := 2n+2m = \dim_{\mathbb{R}}\Gamma\backslash G$. Then for multi-indices $I, J \subset \{1,\ldots,n\}$ and $K, L \subset \{1,\ldots,m\}$, and their complements $I', J' \subset \{1,\ldots,n\}$ and $K', L' \subset \{1,\ldots,m\}$, we have

$$\bar{*}_g\left(x_I \wedge \left(\alpha_J^{-1}\beta_J\right) y_J \wedge \bar{x}_K \wedge \left(\bar{\alpha}_L^{-1}\gamma_L\right) \bar{y}_L\right) = x_{I'} \wedge \left(\alpha_{J'}^{-1}\bar{\beta}_J\right) y_{J'} \wedge \bar{x}_{K'} \wedge \left(\bar{\alpha}_{L'}^{-1}\bar{\gamma}_L\right) \bar{y}_{L'}.$$

Since G is unimodular by the existence of a lattice, [52, Lemma 6.2], we have $\alpha_J\alpha_{J'}\bar{\alpha}_L\bar{\alpha}_{L'} = 1$ and so we have $\beta_{J'}\gamma_{L'} = \beta_J^{-1}\gamma_L^{-1} = \bar{\beta}_J\bar{\gamma}_L^{-1}$. This implies

$$x_{I'} \wedge \left(\alpha_{J'}^{-1}\bar{\beta}_J\right) y_{J'} \wedge \bar{x}_{K'} \wedge \left(\bar{\alpha}_{L'}^{-1}\bar{\gamma}_L\right) \bar{y}_{L'} = x_{I'} \wedge \left(\alpha_{J'}^{-1}\beta_{J'}\right) y_{J'} \wedge \bar{x}_{K'} \wedge \left(\bar{\alpha}_{L'}^{-1}\gamma_{L'}\right) \bar{y}_{L'} \in B_\Gamma^{\bullet,\bullet}.$$

Then we have $\bar{*}_g\left(B_\Gamma^{\bullet,\bullet}\right) \subseteq B_\Gamma^{N-\bullet,N-\bullet}$ and so also

$$\bar{*}_g\left(C_\Gamma^{\bullet,\bullet}\right) \;\subseteq\; C_\Gamma^{N-\bullet,N-\bullet} \;.$$

Hence *(i)*, respectively *(ii)*, follows from Theorem 2.13, respectively Corollary 2.14, and Proposition 2.4.

We consider the sub-complex $A_\Gamma^\bullet \subseteq \wedge^\bullet\Gamma\backslash G \otimes_{\mathbb{R}} \mathbb{C}$ defined in (1). Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^{1,0}$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi(t) = \operatorname{diag}(\alpha_1(t), \ldots, \alpha_m(t))$ for $\alpha_1, \ldots, \alpha_m \in \operatorname{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ characters of $\mathbb{C}^n$. Then, with respect to the basis $\left\{X_1,\,\ldots,\,X_n,\,\bar{X}_1,\,\ldots,\,\bar{X}_n,\,\alpha_1 Y_1,\,\ldots,\,\alpha_m Y_m,\,\bar\alpha_1\bar{Y}_1,\,\ldots,\,\bar\alpha_m\bar{Y}_m\right\}$ of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}^{1,0}\oplus\mathfrak{g}^{0,1}$, we have, for $(t,n)\in G=\mathbb{C}^n\ltimes_\phi N$,

$$
\begin{aligned}
(\mathrm{Ad_s})_{(t,n)} &= \left(\begin{array}{c|c} \mathrm{id}_{(\mathbb{C}^n)^{1,0}\oplus(\mathbb{C}^n)^{0,1}} & 0 \\ \hline 0 & \phi_*\lfloor_{\mathfrak{n}^{1,0}\oplus\mathfrak{n}^{0,1}}(t) \end{array}\right) \\
&= \mathrm{diag}\left(\underbrace{1,\,\ldots,\,1}_{2n\text{ times}},\,\alpha_1(t),\,\ldots,\,\alpha_m(t),\,\bar\alpha_1(t),\,\ldots,\,\bar\alpha_m(t)\right) .
\end{aligned}
$$

Hence we have $J\circ(\mathrm{Ad_s})_{(t,n)}=(\mathrm{Ad_s})_{(t,n)}\circ J$, and we can easily see that $A^{\bullet,\bullet}_\Gamma\subseteq C^{\bullet,\bullet}_\Gamma\subseteq\wedge^{\bullet,\bullet}\Gamma\backslash G$. Since the composition

$$
\frac{\ker \mathrm{d}\lfloor_{A^{p,q}_\Gamma}}{\mathrm{d}\left(A^{p+q-1}_\Gamma\right)} \to \frac{\ker \mathrm{d}\lfloor_{C^{p,q}}}{\mathrm{d}\left(\mathrm{Tot}^{p+q-1}\,C^{\bullet,\bullet}_\Gamma\right)} \to \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\,\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p-q-1}\,\Gamma\backslash G\otimes_{\mathbb{R}}\mathbb{C}\right)}
$$

is an isomorphism, then *(iii)* of the corollary follows. □

Finally we get the following theorem.

Theorem 2.16. *Let $\Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J as in Assumption 2.11. Consider $C^{\bullet,\bullet}_\Gamma$ as in (5). For any $(p,q)\in\mathbb{Z}^2$, the inclusion $C^{\bullet,\bullet}_\Gamma\subseteq\wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the isomorphism*

$$
H\left(C^{p-1,q-1}_\Gamma \stackrel{\partial\overline{\partial}}{\to} C^{p,q}_\Gamma \stackrel{\partial+\overline{\partial}}{\to} C^{p+1,q}_\Gamma\oplus C^{p,q+1}_\Gamma\right) \stackrel{\simeq}{\to} H^{p,q}_{BC}\left(\Gamma\backslash G\right) .
$$

Proof. By Corollary 2.15, the surjectivity follows from Theorem 1.3. The injectivity follows from Proposition 2.2. □

Example 2.17 (The completely-solvable Nakamura manifold, [40, Example 1])**.** The completely-solvable Nakamura manifold, firstly studied by I. Nakamura in [54, page 90], is an example of a cohomologically Kähler non-Kähler solvmanifold, [26], [33, Example 3.1], [27, §3].

Let $G:=\mathbb{C}\ltimes_\phi\mathbb{C}^2$, where

$$
\phi\left(x+\sqrt{-1}\,y\right) := \begin{pmatrix} \mathrm{e}^x & 0 \\ 0 & \mathrm{e}^{-x}\end{pmatrix} \in \mathrm{GL}\left(\mathbb{C}^2\right) .
$$

Then for some $a\in\mathbb{R}$ the matrix $\begin{pmatrix} \mathrm{e}^x & 0 \\ 0 & \mathrm{e}^{-x}\end{pmatrix}$ is conjugate to an element of $\mathrm{SL}(2;\mathbb{Z})$. We have a lattice $\Gamma:=\left(a\,\mathbb{Z}+b\,\sqrt{-1}\,\mathbb{Z}\right)\ltimes_\phi\Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$. Consider the completely-solvable solvmanifold $\Gamma\backslash G$.

(As a matter of notation, we consider holomorphic coordinates $\{z_1,\,z_2,\,z_3\}$, where $\left\{z_1:=x+\sqrt{-1}\,y\right\}$ is the holomorphic coordinate on $\mathbb{C}$, and we shorten, for example, $\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar1}:=\mathrm{e}^{-z_1}\,\mathrm{d}\,z_1\wedge\mathrm{d}\,z_2\wedge\mathrm{d}\,\bar z_1$.)

By A. Hattori's theorem, [37, Corollary 4.2], the de Rham cohomology of $\Gamma\backslash G$ does not depend on Γ and can be computed using just G-left-invariant forms on $\Gamma\backslash G$; more precisely, one gets

$$
\begin{aligned}
H^0_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle 1\right\rangle , \\
H^1_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_1,\;\mathrm{d}\,\bar z_1\right\rangle , \\
H^2_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_{23},\;\mathrm{d}\,z_{1\bar1},\;\mathrm{d}\,z_{2\bar3},\;\mathrm{d}\,z_{3\bar2},\;\mathrm{d}\,z_{\bar2\bar3}\right\rangle , \\
H^3_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_{123},\;\mathrm{d}\,z_{23\bar1},\;\mathrm{d}\,z_{12\bar3},\;\mathrm{d}\,z_{13\bar2},\;\mathrm{d}\,z_{1\bar2\bar3},\;\mathrm{d}\,z_{2\bar1\bar3},\;\mathrm{d}\,z_{3\bar1\bar2},\;\mathrm{d}\,z_{\bar1\bar2\bar3}\right\rangle , \\
H^4_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_{123\bar1},\;\mathrm{d}\,z_{12\bar1\bar3},\;\mathrm{d}\,z_{23\bar2\bar3},\;\mathrm{d}\,z_{13\bar1\bar2},\;\mathrm{d}\,z_{1\bar1\bar2\bar3}\right\rangle , \\
H^5_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_{123\bar2\bar3},\;\mathrm{d}\,z_{23\bar1\bar2\bar3}\right\rangle , \\
H^6_{dR}(\Gamma\backslash G\,;\mathbb{R}) &= \mathbb{R}\left\langle \mathrm{d}\,z_{123\bar1\bar2\bar3}\right\rangle ,
\end{aligned}
$$

where we have listed the harmonic representatives with respect to the G-left-invariant Hermitian metric $g:=\mathrm{d}\,z_1\odot\mathrm{d}\,\bar z_1+\mathrm{e}^{-z_1-\bar z_1}\,\mathrm{d}\,z_2\odot\mathrm{d}\,\bar z_2+\mathrm{e}^{z_1+\bar z_1}\,\mathrm{d}\,z_3\odot\mathrm{d}\,\bar z_3$ instead of their cohomology classes.

We consider $C^{\bullet,\bullet}_\Gamma$ as in (5). The bi-differential bi-graded algebra $B^{\bullet,\bullet}_\Gamma$ varies for a choice of b. By using Theorem 2.16, we compute $H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)\simeq H^{\bullet,\bullet}_{BC}(C^{\bullet,\bullet}_\Gamma)$, case by case:

(i) $b=2m\pi$ for some integer $m\in\mathbb{Z}$;
(ii) $b=(2m+1)\pi$ for some integer $m\in\mathbb{Z}$;

(iii) $b \neq m\pi$ for any integer $m \in \mathbb{Z}$.

Firstly, we write down $C_\Gamma^{\bullet,\bullet}$ case by case in Table 1, Table 2, and Table 3.

case *(i)*	$C_\Gamma^{\bullet,\bullet}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_1,\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_2,\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_3,\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_2,\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_3 \right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{\bar3} \right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13},\ \mathrm{d}\,z_{23},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13} \right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar1},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{2\bar2},\ \mathrm{d}\,z_{2\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar1},\ \mathrm{d}\,z_{3\bar2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{3\bar3},\right.$ $\left.\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{2\bar1},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{1\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{1\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{3\bar1},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{2\bar2},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{3\bar3} \right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{\bar1\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar1\bar3},\ \mathrm{d}\,z_{\bar2\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{\bar1\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{\bar1\bar3} \right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123} \right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar1},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar2},\ \mathrm{d}\,z_{12\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar1},\ \mathrm{d}\,z_{13\bar2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar3},\ \mathrm{d}\,z_{23\bar1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{23\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{23\bar3},\right.$ $\left.\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12\bar1},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13\bar1},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{12\bar2},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{23\bar2},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{13\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{23\bar3} \right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{1\bar1\bar2},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{2\bar1\bar2},\ \mathrm{d}\,z_{3\bar1\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{1\bar1\bar3},\ \mathrm{d}\,z_{2\bar1\bar3},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{3\bar1\bar3},\ \mathrm{d}\,z_{1\bar2\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{2\bar2\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{3\bar2\bar3},\right.$ $\left.\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar1\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar1\bar3},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{2\bar1\bar2},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar2\bar3},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{3\bar1\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar2\bar3} \right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar1\bar2\bar3} \right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{123\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{123\bar3} \right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar1\bar2},\ \mathrm{d}\,z_{12\bar1\bar3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar2\bar3},\ \mathrm{d}\,z_{13\bar1\bar2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar1\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar2\bar3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{23\bar1\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{23\bar1\bar3},\right.$ $\left.\mathrm{d}\,z_{23\bar2\bar3},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{12\bar1\bar2},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{23\bar1\bar2},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12\bar2\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13\bar2\bar3},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{13\bar1\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{23\bar1\bar3} \right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar1\bar2\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{2\bar1\bar2\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{3\bar1\bar2\bar3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar1\bar2\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar1\bar2\bar3} \right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar1\bar2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar1\bar3},\ \mathrm{d}\,z_{123\bar2\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{123\bar1\bar2},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{123\bar1\bar3} \right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar1\bar2\bar3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar1\bar2\bar3},\ \mathrm{d}\,z_{23\bar1\bar2\bar3},\ \mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12\bar1\bar2\bar3},\ \mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13\bar1\bar2\bar3} \right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar1\bar2\bar3} \right\rangle$

TABLE 1. The double complex $C_\Gamma^{\bullet,\bullet}$ for the completely-solvable Nakamura manifold in case *(i)*.

Note that, since $\partial\overline{\partial}\left(C_\Gamma^{\bullet,\bullet}\right) = \{0\}$ for each case, we have, by using Theorem 2.16,

$$H_{BC}^{\bullet,\bullet}(\Gamma\backslash G) \;\simeq\; H_{BC}^{\bullet,\bullet}\left(C_\Gamma^{\bullet,\bullet}\right) \;=\; \ker \mathrm{d}\lfloor_{C_\Gamma^{\bullet,\bullet}} .$$

Hence, we compute the Bott-Chern cohomology of the Nakamura manifold case by case in Table 4 and Table 5; note that, in the case *(iii)*, simply we have:

$$H_{BC}^{\bullet,\bullet}(\Gamma\backslash G) \simeq C_\Gamma^{\bullet,\bullet} \qquad \text{in case } \textit{(iii)}\,. \tag{6}$$

We summarize in Table 6 the results of the computations of the Bott-Chern cohomology as done in Table 4 and Table 5 and (6), and of the Dolbeault cohomology, as done in [40, Example 1].

Remark 2.18. Note that in any case the canonical map $\mathrm{Tot}^\bullet H_{BC}^{\bullet,\bullet}(\Gamma\backslash G) \to H_{dR}^\bullet(\Gamma\backslash G)$ is surjective. (With the notation of [49, 9], this means that, in any case, $\Gamma\backslash G$ is *complex-$\mathcal{C}^\infty$-pure-and-full at every stage*, namely, the de Rham cohomology admits a decomposition in pure-type subgroups with respect to the complex structure.) In the case *(iii)*, by Proposition 1.1, we have $H_{dR}^\bullet(\Gamma\backslash G) \simeq H^\bullet\left(\mathrm{Tot}^\bullet C_\Gamma^{\bullet,\bullet}\right) =$

case *(ii)*	$C_{\Gamma}^{\bullet,\bullet}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1} \right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{1}} \right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23} \right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{1}},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{2\bar{2}},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{2\bar{2}},\ \mathrm{e}^{2z_1}\,\mathrm{d}\, z_{3\bar{3}},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{3\bar{3}},\ \mathrm{d}\, z_{2\bar{3}},\ \mathrm{d}\, z_{3\bar{2}} \right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123} \right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23\bar{1}},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{2}},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{12\bar{2}},\ \mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{3}},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{13\bar{3}},\ \mathrm{d}\, z_{12\bar{3}},\ \mathrm{d}\, z_{13\bar{2}} \right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{2}\bar{3}},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{2\bar{1}\bar{2}},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{2\bar{1}\bar{2}},\ \mathrm{e}^{2z_1}\,\mathrm{d}\, z_{3\bar{1}\bar{3}},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{3\bar{1}\bar{3}},\ \mathrm{d}\, z_{2\bar{1}\bar{3}},\ \mathrm{d}\, z_{3\bar{1}\bar{2}} \right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{1}} \right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{12\bar{1}\bar{3}},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}},\ \mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}},\ \mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}},\ \mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}},\ \mathrm{d}\, z_{23\bar{2}\bar{3}},\ \mathrm{d}\, z_{13\bar{1}\bar{2}} \right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{2}\bar{3}} \right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{1}\bar{2}\bar{3}} \right\rangle$

TABLE 2. The double complex $C_{\Gamma}^{\bullet,\bullet}$ for the completely-solvable Nakamura manifold in case *(ii)*.

$\mathrm{Tot}^{\bullet}\, C_{\Gamma}^{\bullet,\bullet}$ and hence the canonical map $\mathrm{Tot}^{\bullet}\, H_{BC}^{\bullet,\bullet}(\Gamma\backslash G) \to H_{dR}^{\bullet}(\Gamma\backslash G)$ induced by the identity is in fact an isomorphism: this implies that $\Gamma\backslash G$ in case *(iii)* satisfies the $\partial\bar{\partial}$-Lemma (namely, every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact too, see [29]). In [40], it is shown that for some left-invariant Hermitian metric the space of harmonic forms admits the Hodge decomposition and symmetry (see also [41] for higher dimensional examples with the Hodge decomposition and symmetry).

Remark 2.19. In view of [10, Theorem A, Theorem B], stating that, for every compact complex manifold X, for any $k \in \mathbb{Z}$, the inequality

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)\right) \;\geq\; \sum_{p+q=k} \Big(\dim_{\mathbb{C}} H_{\partial}^{p,q}(X) + \dim_{\mathbb{C}} H_{\overline{\partial}}^{p,q}(X)\Big) \;\geq\; 2\,\dim_{\mathbb{C}} H_{dR}^{k}(X;\mathbb{C})$$

holds, and that equalities hold for any $k \in \mathbb{Z}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma, one gets that the non-negative integer numbers $\sum_{p+q=k}\left(\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)\right) - 2\,\dim_{\mathbb{C}} H_{dR}^{k}(X;\mathbb{C}) \in \mathbb{N}$, varying $k \in \mathbb{Z}$, provide a "measure" of the non-Kählerianity of X.

Note that, for the completely-solvable Nakamura manifold, in any case, one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X) \;=\; \dim_{\mathbb{C}} H_{\partial}^{p,q}(X) + \dim_{\mathbb{C}} H_{\overline{\partial}}^{p,q}(X)$$

for any $(p,q) \in \mathbb{Z}^2$. On the other hand,

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)\right) - 2\,\dim_{\mathbb{C}} H_{dR}^{k}(X;\mathbb{C}) \;=\; \begin{cases} 8 & \text{for } k \in \{1,\,5\} \\ 20 & \text{for } k \in \{2,\,4\} \\ 24 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case } (i)\,,$$

case *(iii)*	$C_\Gamma^{\bullet,\bullet}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1} \right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{1}} \right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23} \right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{1}},\ \mathrm{d}\, z_{2\bar{3}},\ \mathrm{d}\, z_{3\bar{2}} \right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123} \right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23\bar{1}},\ \mathrm{d}\, z_{12\bar{3}},\ \mathrm{d}\, z_{13\bar{2}} \right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{2}\bar{3}},\ \mathrm{d}\, z_{2\bar{1}\bar{3}},\ \mathrm{d}\, z_{3\bar{1}\bar{2}} \right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{1}} \right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{12\bar{1}\bar{3}},\ \mathrm{d}\, z_{23\bar{2}\bar{3}},\ \mathrm{d}\, z_{13\bar{1}\bar{2}} \right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{1\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{2}\bar{3}} \right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{23\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\, z_{123\bar{1}\bar{2}\bar{3}} \right\rangle$

TABLE 3. The double complex $C_\Gamma^{\bullet,\bullet}$ for the completely-solvable Nakamura manifold in case *(iii)*.

and

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) - 2\, \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;=\; \begin{cases} 0 & \text{for } k \in \{1,\, 5\} \\ 4 & \text{for } k \in \{2,\, 4\} \\ 8 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case } \textit{(ii)}\,,$$

and

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) - 2\, \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;=\; \begin{cases} 0 & \text{for } k \in \{1,\, 5\} \\ 0 & \text{for } k \in \{2,\, 4\} \\ 0 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case } \textit{(iii)}\,.$$

In particular, by [10, Theorem B], one gets that $\Gamma\backslash G$ in case *(iii)* satisfies the $\partial\overline{\partial}$-Lemma, as noticed also in Remark 2.18.

Given a property depending on the complex structure, one says that it is *open under small deformations* (respectively, *strongly-closed under small deformations*) if, for any complex-analytic families of compact complex manifolds parametrized by $\mathcal{B}$, the set of parameters for which the property holds is open (respectively, closed) in the strong topology of $\mathcal{B}$.

We recall that satisfying the $\partial\overline{\partial}$-Lemma is an open property under small deformations, see [70, Proposition 9.21], [73, Theorem 5.12], [65, §B], [10, Corollary 2.7]. On the other hand, as pointed out by Luis Ugarte, the completely-solvable Nakamura manifold provides a counterexample to the strongly-closedness of the property of satisfying the $\partial\overline{\partial}$-Lemma: indeed, complex structures in class *(iii)* satisfy the $\partial\overline{\partial}$-Lemma while complex structures in classes *(i)* and *(ii)* do not. We have hence the following theorem.

Theorem 2.20. *Satisfying the $\partial\overline{\partial}$-Lemma is not a strongly-closed property under small deformations of the complex structure.*

case *(i)*	$H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1}]\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{1}}]\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{12}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{13}],\ [\mathrm{d}\, z_{23}]\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1\bar{1}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{1\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{1\bar{3}}],\ [\mathrm{d}\, z_{2\bar{3}}],\ [\mathrm{d}\, z_{3\bar{2}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{2\bar{1}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{3\bar{1}}]\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123}]\right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{12\bar{1}}],\ [\mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{2}}],\ [\mathrm{d}\, z_{12\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{13\bar{1}}],\ [\mathrm{d}\, z_{13\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{3}}],\ [\mathrm{d}\, z_{23\bar{1}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{12\bar{1}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{13\bar{1}}]\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{1\bar{1}\bar{2}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{2\bar{1}\bar{2}}],\ [\mathrm{d}\, z_{3\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{1\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{2\bar{1}\bar{3}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{3\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{1\bar{2}\bar{3}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{1\bar{1}\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{1\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123\bar{1}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{123\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{123\bar{3}}]\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}}],\ [\mathrm{d}\, z_{12\bar{1}\bar{3}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{12\bar{2}\bar{3}}],\ [\mathrm{d}\, z_{13\bar{1}\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{13\bar{2}\bar{3}}],\ [\mathrm{d}\, z_{23\bar{2}\bar{3}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{23\bar{1}\bar{2}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{23\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{2\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{3\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{123\bar{1}\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{123\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{123\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{123\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{123\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\, z_{13\bar{1}\bar{2}\bar{3}}],\ [\mathrm{d}\, z_{23\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\, z_{13\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123\bar{1}\bar{2}\bar{3}}]\right\rangle$

Table 4. The Bott-Chern cohomology of the completely-solvable Nakamura manifold in case *(i)*.

Remark 2.21. Actually, as remarked by Luis Ugarte, in defining closedness for deformations, one usually considers the Zariski topology, see, e.g., [56]: namely, a property $\mathcal{P}$ is said to be *(Zariski-)closed* if, for any family $\{X_t\}_{t\in\Delta}$ of compact complex manifolds such that $\mathcal{P}$ holds for any $t \in \Delta \setminus \{0\}$ in the punctured-disk, then $\mathcal{P}$ holds also for X_0. In [7], a family of deformations of the complex parallelizable Nakamura manifold is studied in order to prove that satisfying the $\partial\overline{\partial}$-Lemma is also non-(Zariski-)closed.

2.4.2. *Complex parallelizable solvmanifolds.* Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ, and denote by $2n$ the real dimension of G. Denote the Lie algebra naturally associated to G by $\mathfrak{g}$. We use the following lemma.

Lemma 2.22. *Let α, β be holomorphic characters of a connected simply-connected complex solvable Lie group G. If $\alpha\bar{\beta}$ is a unitary character, then $\alpha = \beta^{-1}$.*

Proof. Since we have $\alpha([G,G]) = [\alpha(G), \alpha(G)] = 1$ and $\beta([G,G]) = [\beta(G), \beta(G)] = 1$, we can regard α and β as characters of $G/[G,G]$. Since G is connected simply-connected, $G/[G,G]$ is also connected simply-connected, see [28, Theorem 3.5]. Since $G/[G,G]$ is Abelian, it is sufficient to show the lemma in the case $G = \mathbb{C}^n$. For the coordinate set $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$, we write $\alpha = \exp\left(\sum_{j=1}^n a_j z_j\right)$ and $\beta = \exp\left(\sum_{j=1}^n b_j z_j\right)$, for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$. If $\alpha\bar{\beta}$ is unitary, then we have $\Re\left(\sum_{j=1}^n \left(a_j z_j + \bar{b}_j \bar{z}_j\right)\right) = 0$. By simple computations, we have $a_j = -b_j$ for any $j \in \{1, \ldots, n\}$. Hence the lemma follows. □

Denote by $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) the Lie algebra of the G-left-invariant holomorphic (respectively, anti-holomorphic) vector fields on G. As a (real) Lie algebra, we have an isomorphism $\mathfrak{g}_+ \simeq \mathfrak{g}_-$ by means of the complex conjugation.

case *(ii)*	$H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1}]\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{1}}]\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{23}]\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1\bar{1}}],\ [\mathrm{d}\, z_{2\bar{3}}],\ [\mathrm{d}\, z_{3\bar{2}}]\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123}]\right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{23\bar{1}}],\ [\mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{3}}],\ [\mathrm{d}\, z_{12\bar{3}}],\ [\mathrm{d}\, z_{13\bar{2}}]\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1\bar{2}\bar{3}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{2\bar{1}\bar{2}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{3\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{2\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{3\bar{1}\bar{2}}]\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123\bar{1}}]\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{12\bar{1}\bar{3}}],\ [\mathrm{e}^{-2z_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\, z_{12\bar{1}\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\, z_{13\bar{1}\bar{3}}],\ [\mathrm{d}\, z_{23\bar{2}\bar{3}}],\ [\mathrm{d}\, z_{13\bar{1}\bar{2}}]\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{1\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{23\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\, z_{123\bar{1}\bar{2}\bar{3}}]\right\rangle$

Table 5. The Bott-Chern cohomology of the completely-solvable Nakamura manifold in case *(ii)*.

Let N be the nilradical of G. We can take a connected simply-connected complex nilpotent subgroup $C \subseteq G$ such that $G = C \cdot N$, see, e.g., [28, Proposition 3.3]. Since C is nilpotent, the map

$$C \ni c \mapsto (\mathrm{Ad}_c)_{\mathrm{s}} \in \mathrm{Aut}(\mathfrak{g}_+)$$

is a homomorphism, where $(\mathrm{Ad}_c)_{\mathrm{s}}$ is the semi-simple part of the Jordan decomposition of Ad_c. Let $\mathfrak{c}$ be the Lie algebra of C; we take a subspace $V \subseteq \mathfrak{c}$ such that $\mathfrak{g} = V \oplus \mathfrak{n}$. Then the diagonalizable representation Ad_{s} constructed above, §2.4, is identified with the map

$$G = C \cdot N \ni c \cdot n \mapsto (\mathrm{Ad}_c)_s \in \mathrm{Aut}(\mathfrak{g}),$$

see [43, Remark 4].

We have a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}_+$ such that, for $c \in C$,

$$(\mathrm{Ad}_c)_{\mathrm{s}} \;=\; \mathrm{diag}\left(\alpha_1(c), \ldots, \alpha_n(c)\right) \;,$$

for some characters $\alpha_1, \ldots, \alpha_n$ of C. By $G = C \cdot N$, we have $G/N = C/C \cap N$ and regard $\alpha_1, \ldots, \alpha_n$ as characters of G. Let $\{x_1, \ldots, x_n\}$ be the basis of $\mathfrak{g}_+^*$ which is dual to $\{X_1, \ldots, X_n\}$.

Theorem 2.23. ([43, Corollary 6.2 and its proof]) *Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ. Denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Consider a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}_+$ of the G-left-invariant holomorphic vector fields on G with respect to which $(\mathrm{Ad}_c)_{\mathrm{s}} = \mathrm{diag}\left(\alpha_1(c), \ldots, \alpha_n(c)\right)$ for some characters $\alpha_1, \ldots, \alpha_n$ of C. Regard $\alpha_1, \ldots, \alpha_n$ as characters of G. Let $B^\bullet_\Gamma$ be the sub-complex of $\left(\wedge^{0,\bullet} \Gamma\backslash G \,,\, \overline{\partial}\right)$ defined as*

$$B^\bullet_\Gamma \;:=\; \left\langle \frac{\bar\alpha_I}{\alpha_I}\, \bar{x}_I \;\middle|\; I \subseteq \{1, \ldots, n\} \text{ such that } \left.\left(\frac{\bar\alpha_I}{\alpha_I}\right)\right\lfloor_\Gamma = 1 \right\rangle \;, \tag{7}$$

(where we shorten, e.g., $\alpha_I := \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_k}$ for a multi-index $I = (i_1, \ldots, i_k)$). Then the inclusion $B^\bullet_\Gamma \hookrightarrow \wedge^{0,\bullet} \Gamma\backslash G$ induces the isomorphism

$$H^\bullet\left(B^\bullet_\Gamma, \overline{\partial}\right) \stackrel{\simeq}{\to} H^{0,\bullet}_{\overline{\partial}}(\Gamma\backslash G) \;.$$

	dR	case *(i)* $\bar\partial$	case *(i)* BC	case *(ii)* $\bar\partial$	case *(ii)* BC	case *(iii)* $\bar\partial$	case *(iii)* BC
$(\mathbf{0},\mathbf{0})$	1	1	1	1	1	1	1
$(\mathbf{1},\mathbf{0})$	2	3	1	1	1	1	1
$(\mathbf{0},\mathbf{1})$		3	1	1	1	1	1
$(\mathbf{2},\mathbf{0})$	5	3	3	1	1	1	1
$(\mathbf{1},\mathbf{1})$		9	7	5	3	3	3
$(\mathbf{0},\mathbf{2})$		3	3	1	1	1	1
$(\mathbf{3},\mathbf{0})$	8	1	1	1	1	1	1
$(\mathbf{2},\mathbf{1})$		9	9	5	5	3	3
$(\mathbf{1},\mathbf{2})$		9	9	5	5	3	3
$(\mathbf{0},\mathbf{3})$		1	1	1	1	1	1
$(\mathbf{3},\mathbf{1})$	5	3	3	1	1	1	1
$(\mathbf{2},\mathbf{2})$		9	11	5	7	3	3
$(\mathbf{1},\mathbf{3})$		3	3	1	1	1	1
$(\mathbf{3},\mathbf{2})$	2	3	5	1	1	1	1
$(\mathbf{2},\mathbf{3})$		3	5	1	1	1	1
$(\mathbf{3},\mathbf{3})$	1	1	1	1	1	1	1

Table 6. The dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the completely-solvable Nakamura manifold.

By this theorem, since $\Gamma\backslash G$ is complex parallelizable, for the differential bi-graded algebra $\left(\wedge^{\bullet}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{\bullet},\, \bar\partial\right)$, the inclusion $\wedge^{\bullet_1}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{\bullet_2}\hookrightarrow \wedge^{\bullet_1,\bullet_2}\,\Gamma\backslash G$ induces the isomorphism

$$\wedge^{\bullet_1}\mathfrak{g}_+^*\otimes_{\mathbb{C}} H^{\bullet_2}_{\bar\partial}(B_\Gamma^{\bullet}) \stackrel{\simeq}{\to} H^{\bullet_1,\bullet_2}_{\bar\partial}(\Gamma\backslash G)\,.$$

Consider the G-left-invariant Hermitian metric

$$g \;:=\; \sum_{j=1}^{n} x_j\odot \bar x_j\,.$$

Then, for $x_I\wedge\frac{\bar\alpha_K}{\alpha_K}\,\bar x_K\in\wedge^{|I|}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{|K|}$, since G is unimodular, [52, Lemma 6.2], we have

$$\bar *_g\left(x_I\wedge\frac{\bar\alpha_K}{\alpha_K}\,\bar x_K\right) \;=\; x_{I'}\wedge\frac{\alpha_K}{\bar\alpha_K}\,\bar x_{K'} \;=\; x_{I'}\wedge\frac{\bar\alpha_{K'}}{\alpha_{K'}}\,\bar x_{K'} \;\in\; \wedge^{n-|I|}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{n-|K|}\,,$$

where $I':=\{1,\ldots,n\}\setminus I$ and $K':=\{1,\ldots,n\}\setminus K$ are the complements of I and K respectively. Hence we have $\bar *_g(\wedge^{\bullet}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{\bullet})\subseteq \wedge^{n-\bullet}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{n-\bullet}$.

We consider the space

$$\bar B_\Gamma^{\bullet} \;=\; \left\langle \frac{\alpha_I}{\bar\alpha_I}\,x_I \;\middle|\; I\subseteq\{1,\ldots,n\} \text{ such that } \left(\frac{\alpha_I}{\bar\alpha_I}\right)\Big\lfloor_{\Gamma} = 1\right\rangle\,.$$

Then the inclusion $\bar B_\Gamma^{\bullet_1}\otimes_{\mathbb{C}}\wedge^{\bullet_2}\mathfrak{g}_-^*\subseteq\wedge^{\bullet_1,\bullet_2}\,\Gamma\backslash G$ induces the isomorphism in ∂-cohomology

$$H^{\bullet_1}\left(\bar B_\Gamma^{\bullet}\otimes_{\mathbb{C}}\wedge^{\bullet_2}\mathfrak{g}_-^*,\,\partial\right)\stackrel{\simeq}{\to} H^{\bullet_1,\bullet_2}_{\partial}\left(\Gamma\backslash G\right)\,.$$

Consider

$$C^{\bullet_1,\bullet_2} \;:=\; \wedge^{\bullet_1}\mathfrak{g}_+^*\otimes_{\mathbb{C}} B_\Gamma^{\bullet_2}+\bar B_\Gamma^{\bullet_1}\otimes_{\mathbb{C}}\wedge^{\bullet_2}\mathfrak{g}_-^*\,. \tag{8}$$

Then we have $\bar *_g\left(C^{\bullet_1,\bullet_2}\right)\subseteq C^{n-\bullet_1,n-\bullet_2}$.

As similar to Corollary 2.15, we can show the following result.

Corollary 2.24. *Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ. Denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Consider the sub-complex $C_\Gamma^{\bullet,\bullet} \subseteq \wedge^{\bullet,\bullet}\Gamma\backslash G$ as defined in* (8).

(i) The inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the ∂-cohomology isomorphism

$$H^{\bullet,\bullet}(C_\Gamma^{\bullet,\bullet}, \partial) \stackrel{\simeq}{\to} H_\partial^{\bullet,\bullet}(\Gamma\backslash G) .$$

(ii) The inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the $\overline{\partial}$-cohomology isomorphism

$$H^{\bullet,\bullet}(C_\Gamma^{\bullet,\bullet}, \overline{\partial}) \stackrel{\simeq}{\to} H_{\overline{\partial}}^{\bullet,\bullet}(\Gamma\backslash G) .$$

(iii) The inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces, for any $(p,q) \in \mathbb{Z}^2$, the surjection

$$\frac{\ker \mathrm{d}\lfloor_{C^{p,q}}}{\mathrm{d}\left(\mathrm{Tot}^{p+q-1} C_\Gamma^{\bullet,\bullet}\right)} \to \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\Gamma\backslash G \otimes_\mathbb{R} \mathbb{C}\right)} .$$

Proof. By $\bar{*}_g\left(C^{\bullet_1,\bullet_2}\right) \subseteq C^{n-\bullet_1,n-\bullet_2}$, the first and second assertions follow as similar to the proof of Corollary 2.15.

By denoting the complex structure by J, for any $c \in C$, since we have $\mathrm{Ad}_c \circ J = J \circ \mathrm{Ad}_c$, we have $(\mathrm{Ad}_c)_\mathrm{s} \circ J = J \circ (\mathrm{Ad}_c)_\mathrm{s}$, and hence we have $(\mathrm{Ad}_\mathrm{s})_g \circ J = J \circ (\mathrm{Ad}_\mathrm{s})_g$ for any $g \in G$. We consider the sub-complex $A_\Gamma^\bullet \subseteq \wedge^\bullet \Gamma\backslash G \otimes_\mathbb{R} \mathbb{C}$ as in (1), see Theorem 2.8. By Corollary 2.10, the inclusion $A_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{p,q}\Gamma\backslash G$ induces the isomorphism

$$\frac{\ker \mathrm{d}\lfloor_{A_\Gamma^{p,q}}}{\mathrm{d}\left(A_\Gamma^{p+q-1}\right)} \stackrel{\simeq}{\to} \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p+q-1}\Gamma\backslash G \otimes_\mathbb{R} \mathbb{C}\right)} .$$

We have

$$A_\Gamma^\bullet = \langle \alpha_I\,\bar{\alpha}_J\,x_I \wedge \bar{x}_J \mid I, J \subseteq \{1,\ldots,n\} \text{ such that } (\alpha_I\,\bar{\alpha}_J)\lfloor_\Gamma = 1\rangle .$$

For $(\alpha_I\,\bar{\alpha}_J)\lfloor_\Gamma = 1$, since we can regard $\alpha_I\,\bar{\alpha}_J$ as a function on $\Gamma\backslash G$, the image of $\alpha_I\,\bar{\alpha}_J$ is compact and hence it is unitary. By Lemma 2.22, we have $\alpha_I\,\bar{\alpha}_J = \frac{\bar{\alpha}_J}{\alpha_J}$. Hence we have the inclusion $A_\Gamma^\bullet \subseteq \mathrm{Tot}^\bullet \wedge^\bullet \mathfrak{g}_+^* \otimes B_\Gamma^\bullet$ and so we have the inclusion $A_\Gamma^{\bullet,\bullet} \subseteq C_\Gamma^{\bullet,\bullet} \subseteq \wedge^{\bullet,\bullet}\Gamma\backslash G$. Since the composition

$$\frac{\ker \mathrm{d}\lfloor_{A_\Gamma^{p,q}}}{\mathrm{d}\left(A_\Gamma^{p+q-1}\right)} \to \frac{\ker \mathrm{d}\lfloor_{C^{p,q}}}{\mathrm{d}\left(\mathrm{Tot}^{p+q-1} C_\Gamma^{\bullet,\bullet}\right)} \to \frac{\ker \mathrm{d}\lfloor_{\wedge^{p,q}\Gamma\backslash G}}{\mathrm{d}\left(\wedge^{p-q-1}\Gamma\backslash G\right)}$$

is an isomorphism, then the third assertion of the corollary follows. □

By this, we get the following result.

Theorem 2.25. *Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ. Consider the sub-complex $C_\Gamma^{\bullet,\bullet} \subseteq \wedge^{\bullet,\bullet}\Gamma\backslash G$ as defined in* (8). *The inclusion $C_\Gamma^{\bullet,\bullet} \hookrightarrow \wedge^{\bullet,\bullet}\Gamma\backslash G$ induces the isomorphism*

$$H\left(C_\Gamma^{\bullet-1,\bullet-1} \stackrel{\partial\overline{\partial}}{\to} C_\Gamma^{\bullet,\bullet} \stackrel{\mathrm{d}}{\to} C_\Gamma^{\bullet+1,\bullet} \oplus C_\Gamma^{\bullet,\bullet+1}\right) \stackrel{\simeq}{\to} H_{BC}^{\bullet,\bullet}(\Gamma\backslash G) .$$

Example 2.26 (The complex parallelizable Nakamura manifold)**.** Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ be such that

$$\phi(z) = \begin{pmatrix} \mathrm{e}^z & 0 \\ 0 & \mathrm{e}^{-z} \end{pmatrix} .$$

Then there exist $a + \sqrt{-1}\,b \in \mathbb{C}$ and $c + \sqrt{-1}\,d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}\,b) + \mathbb{Z}(c + \sqrt{-1}\,d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}\,b)$ and $\phi(c + \sqrt{-1}\,d)$ are conjugate to elements of $\mathrm{SL}(4;\mathbb{Z})$, where we regard $\mathrm{SL}(2;\mathbb{C}) \subset \mathrm{SL}(4;\mathbb{R})$, see [36]. Hence we have a lattice $\Gamma := \left(\mathbb{Z}\left(a + \sqrt{-1}\,b\right) + \mathbb{Z}\left(c + \sqrt{-1}\,d\right)\right) \ltimes_\phi \Gamma''$ of G such that Γ'' is a lattice of $\mathbb{C}^2$. Let $X := \Gamma\backslash G$ be the *complex parallelizable Nakamura manifold*, [54, §2].

We take the connected simply-connected complex nilpotent subgroup $C := \mathbb{C} \subseteq G$ such that $G = C \cdot N$, where N is the nilradical of G. Recall that $\mathfrak{g}_+$ denotes the Lie algebra of the G-left-invariant holomorphic vector fields on G. For a coordinate set (z_1, z_2, z_3) of $\mathbb{C} \ltimes_\phi \mathbb{C}^2$, we have the basis $\left\{\frac{\partial}{\partial z_1},\ \mathrm{e}^{z_1}\,\frac{\partial}{\partial z_2},\ \mathrm{e}^{-z_1}\,\frac{\partial}{\partial z_3}\right\}$ of $\mathfrak{g}_+$ such that

$$\left(\mathrm{Ad}_{(z_1,z_2,z_3)}\right)_\mathrm{s} = \mathrm{diag}\left(1,\, \mathrm{e}^{z_1},\, \mathrm{e}^{-z_1}\right) \in \mathrm{Aut}(\mathfrak{g}_+) .$$

(a) If $b \in \pi\,\mathbb{Z}$ and $d \in \pi\,\mathbb{Z}$, then, for $z \in \left(a+\sqrt{-1}\,b\right)\,\mathbb{Z}+\left(c+\sqrt{-1}\,d\right)\,\mathbb{Z}$, we have $\phi(z) \in \mathrm{SL}(2;\mathbb{R})$. Since $\left(\frac{\mathrm{e}^{z_1}}{\mathrm{e}^{\bar z_1}}\right)\lfloor_\Gamma = (\mathrm{e}^{z_1-\bar z_1})\lfloor_\Gamma = 1$, we have

$$B^\bullet_\Gamma \;=\; \wedge^\bullet\mathbb{C}\left\langle \mathrm{d}\,z_{\bar 1},\; \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar 2},\; \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar 3}\right\rangle \;.$$

Hence the double complex $C^{\bullet,\bullet}_\Gamma$ in case *(a)* is the one in Table 7. (We recall that, in order to shorten the notation, we write, for example, $\mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{1\bar 3} := \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_1\wedge\mathrm{d}\,\bar z_3$.)

case *(a)*	$C^{\bullet,\bullet}_\Gamma$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_1,\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_2,\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_3,\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_2,\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_3\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar 1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{\bar 3}\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13},\ \mathrm{d}\,z_{23},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{12},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{13}\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar 1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar 3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 1},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{2\bar 2},\ \mathrm{d}\,z_{2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 1},\ \mathrm{d}\,z_{3\bar 2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{3\bar 3},\right.$ $\left.\mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{2\bar 1},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{1\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{3\bar 1},\ \mathrm{e}^{-2\bar z_1}\,\mathrm{d}\,z_{2\bar 2},\ \mathrm{e}^{2\bar z_1}\,\mathrm{d}\,z_{3\bar 3}\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{\bar 1\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{\bar 1\bar 3},\ \mathrm{d}\,z_{\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{\bar 1\bar 3}\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123}\right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 1},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar 2},\ \mathrm{d}\,z_{12\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 1},\ \mathrm{d}\,z_{13\bar 2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar 3},\ \mathrm{d}\,z_{23\bar 1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{23\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{23\bar 3},\right.$ $\left.\mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{12\bar 1},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{13\bar 1},\ \mathrm{e}^{-2\bar z_1}\,\mathrm{d}\,z_{12\bar 2},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{23\bar 2},\ \mathrm{e}^{2\bar z_1}\,\mathrm{d}\,z_{13\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{23\bar 3}\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{1\bar 1\bar 2},\ \mathrm{e}^{-2\bar z_1}\,\mathrm{d}\,z_{2\bar 1\bar 2},\ \mathrm{d}\,z_{3\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{1\bar 1\bar 3},\ \mathrm{d}\,z_{2\bar 1\bar 3},\ \mathrm{e}^{2\bar z_1}\,\mathrm{d}\,z_{3\bar 1\bar 3},\ \mathrm{d}\,z_{1\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{2\bar 2\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{3\bar 2\bar 3},\right.$ $\left.\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar 1\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar 1\bar 3},\ \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{2\bar 1\bar 2},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 2\bar 3},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{3\bar 1\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 2\bar 3}\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar 1\bar 2\bar 3}\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar 1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{123\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{123\bar 3}\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar 1\bar 2},\ \mathrm{d}\,z_{12\bar 1\bar 3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 2\bar 3},\ \mathrm{d}\,z_{13\bar 1\bar 2},\ \mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar 1\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 2\bar 3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{23\bar 1\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{23\bar 1\bar 3},\right.$ $\left.\mathrm{d}\,z_{23\bar 2\bar 3},\ \mathrm{e}^{-2\bar z_1}\,\mathrm{d}\,z_{12\bar 1\bar 2},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{23\bar 1\bar 2},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{12\bar 2\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{13\bar 2\bar 3},\ \mathrm{e}^{2\bar z_1}\,\mathrm{d}\,z_{13\bar 1\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{23\bar 1\bar 3}\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar 1\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{2\bar 1\bar 2\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{3\bar 1\bar 2\bar 3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 1\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 1\bar 2\bar 3}\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar 1\bar 2},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar 1\bar 3},\ \mathrm{d}\,z_{123\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{123\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{123\bar 1\bar 3}\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 1\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 1\bar 2\bar 3},\ \mathrm{d}\,z_{23\bar 1\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{12\bar 1\bar 2\bar 3},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{13\bar 1\bar 2\bar 3}\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar 1\bar 2\bar 3}\right\rangle$

TABLE 7. The double complex $C^{\bullet,\bullet}_\Gamma$ in (8) for the complex parallelizable Nakamura manifold in case *(a)*.

We compute the Bott-Chern cohomology for the complex parallelizable Nakamura manifold in case *(a)* in Table 8.

The differential algebra $A^\bullet_\Gamma$ for the complex parallelizable Nakamura manifold in case *(a)* is summarized in Table 9.

Remark 2.27. Suppose $b \in 2\pi\,\mathbb{Z}$ and $d \in 2\pi\,\mathbb{Z}$. Considering another Lie group $H := \mathbb{C}\ltimes_\psi\mathbb{C}^2$ such that

$$\psi(z) \;:=\; \begin{pmatrix} \mathrm{e}^{\frac{1}{2}(z_1+\bar z_1)} & 0 \\ 0 & \mathrm{e}^{-\frac{1}{2}(z_1+\bar z_1)} \end{pmatrix},$$

the correspondence $G \in (z_1,\, z_2,\, z_3) \mapsto (z_1,\, z_2,\, z_3) \in H$ gives an embedding $\Gamma \hookrightarrow H$ as a lattice and hence we can identify $\Gamma\backslash G$ with $\Gamma\backslash H$, see [74, Section 3]. Since H is equal to the solvable completely-solvable Lie group in Example 2.17, this case is identified with case *(i)* in Example 2.17. Note that $A^\bullet_\Gamma$ is not G-left-invariant in this case (for example the 2-form $\mathrm{d}\,z_{2\bar 3}$ is not G-left-invariant)

case *(a)*	$H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{1}]\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{\bar{1}}]\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13}],\ [\mathrm{d}\,z_{23}]\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{1\bar{1}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar{3}}],\ [\mathrm{d}\,z_{2\bar{3}}],\ [\mathrm{d}\,z_{3\bar{2}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{2\bar{1}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{3\bar{1}}]\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{123}]\right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\Big\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{1}}],\ [\mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar{2}}],\ [\mathrm{d}\,z_{12\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{1}}],\ [\mathrm{d}\,z_{13\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar{3}}],$
	$[\mathrm{d}\,z_{23\bar{1}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12\bar{1}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13\bar{1}}]\Big\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\Big\langle [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{1\bar{1}\bar{2}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{2\bar{1}\bar{2}}],\ [\mathrm{d}\,z_{3\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{1\bar{1}\bar{3}}],\ [\mathrm{d}\,z_{2\bar{1}\bar{3}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{3\bar{1}\bar{3}}],$
	$[\mathrm{d}\,z_{1\bar{2}\bar{3}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{1\bar{1}\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{1\bar{1}\bar{3}}]\Big\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{123\bar{1}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar{3}}]\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\Big\langle [\mathrm{e}^{-2z_1}\,\mathrm{d}\,z_{12\bar{1}\bar{2}}],\ [\mathrm{d}\,z_{12\bar{1}\bar{3}}],\ [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{2}\bar{3}}],\ [\mathrm{d}\,z_{13\bar{1}\bar{2}}],\ [\mathrm{e}^{2z_1}\,\mathrm{d}\,z_{13\bar{1}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{2}\bar{3}}],$
	$[\mathrm{d}\,z_{23\bar{2}\bar{3}}],\ [\mathrm{e}^{-2\bar{z}_1}\,\mathrm{d}\,z_{12\bar{1}\bar{2}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{23\bar{1}\bar{2}}],\ [\mathrm{e}^{2\bar{z}_1}\,\mathrm{d}\,z_{13\bar{1}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{23\bar{1}\bar{3}}]\Big\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{1\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{2\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{3\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{123\bar{1}\bar{2}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{123\bar{1}\bar{3}}],\ [\mathrm{d}\,z_{123\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{123\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{123\bar{1}\bar{3}}]\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{1}\bar{2}\bar{3}}],\ [\mathrm{d}\,z_{23\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{12\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{13\bar{1}\bar{2}\bar{3}}]\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle [\mathrm{d}\,z_{123\bar{1}\bar{2}\bar{3}}]\right\rangle$

TABLE 8. The Bott-Chern cohomology of the complex parallelizable Nakamura manifold in case *(a)*.

case *(a)*	$A^{\bullet}_{\Gamma}$
0	$\mathbb{C}\left\langle 1\right\rangle$
1	$\mathbb{C}\left\langle \mathrm{d}\,z_{1},\ \mathrm{d}\,z_{\bar{1}}\right\rangle$
2	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar{1}},\ \mathrm{d}\,z_{23},\ \mathrm{d}\,z_{2\bar{3}},\ \mathrm{d}\,z_{3\bar{2}},\ \mathrm{d}\,z_{\bar{2}\bar{3}}\right\rangle$
3	$\mathbb{C}\left\langle \mathrm{d}\,z_{123},\ \mathrm{d}\,z_{12\bar{3}},\ \mathrm{d}\,z_{13\bar{2}},\ \mathrm{d}\,z_{3\bar{1}\bar{2}},\ \mathrm{d}\,z_{2\bar{1}\bar{3}},\ \mathrm{d}\,z_{\bar{1}\bar{2}\bar{3}},\ \mathrm{d}\,z_{\bar{1}23},\ \mathrm{d}\,z_{1\bar{2}\bar{3}}\right\rangle$
4	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar{1}},\ \mathrm{d}\,z_{13\bar{1}\bar{2}},\ \mathrm{d}\,z_{23\bar{2}\bar{3}},\ \mathrm{d}\,z_{12\bar{1}\bar{3}},\ \mathrm{d}\,z_{1\bar{1}\bar{2}\bar{3}}\right\rangle$
5	$\mathbb{C}\left\langle \mathrm{d}\,z_{23\bar{1}\bar{2}\bar{3}},\ \mathrm{d}\,z_{123\bar{2}\bar{3}}\right\rangle$
6	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar{1}\bar{2}\bar{3}}\right\rangle$

TABLE 9. The cochain complex $A^{\bullet}_{\Gamma}$ in (1) for the complex parallelizable Nakamura manifold in case *(a)*.

and hence $H^{\bullet}\left(\wedge^{\bullet}\mathfrak{g}^{*},\,\mathrm{d}\right) \not\simeq H^{\bullet}_{dR}\left(\Gamma\backslash G\,;\mathbb{R}\right)$, see also [27, Corollary 4.2]. On the other hand, we have $H^{\bullet}\left(\wedge^{\bullet}\mathfrak{h}^{*},\,\mathrm{d}\right) \simeq H^{\bullet}_{dR}\left(\Gamma\backslash H\,;\mathbb{R}\right)$, where $\mathfrak{h}$ is the Lie algebra of H. In [23, Main Theorem], it is proven that, for any solvmanifold $\Gamma\backslash G$, there exist a connected simply-connected solvable Lie group $\tilde{G}$ and a finite index subgroup $\tilde{\Gamma} \subseteq \Gamma$ such that $H^{\bullet}\left(\wedge^{\bullet}\tilde{\mathfrak{g}}^{*},\,\mathrm{d}\right) \simeq H^{\bullet}_{dR}\left(\tilde{\Gamma}\Big\backslash G\,;\mathbb{R}\right)$, where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$.

(b) If $b \not\in \pi\,\mathbb{Z}$ or $d \not\in \pi\,\mathbb{Z}$, then the sub-complex $B^{\bullet}_{\Gamma}$ defined in (7) is

$$B^{1}_{\Gamma} \;=\; \mathbb{C}\left\langle \mathrm{d}\,\bar{z}_1\right\rangle\ ,$$

$$B^{2}_{\Gamma} \;=\; \mathbb{C}\left\langle \mathrm{d}\,\bar{z}_2 \wedge \mathrm{d}\,\bar{z}_3\right\rangle\ ,$$

$$B^3_\Gamma \;=\; \mathbb{C}\left\langle \mathrm{d}\,\bar z_1 \wedge \mathrm{d}\,\bar z_2 \wedge \mathrm{d}\,\bar z_3 \right\rangle \;.$$

Then the double complex $C^{\bullet,\bullet}_\Gamma$ is given in Table 10.

case *(b)*	$C^{\bullet,\bullet}_\Gamma$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_1,\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_2,\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_3 \right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar 1},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,\bar z_3 \right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13},\ \mathrm{d}\,z_{23} \right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar 1},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 1},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 1},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{1\bar 3} \right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{\bar 1\bar 3},\ \mathrm{d}\,z_{\bar 2\bar 3} \right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123} \right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 1},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 1},\ \mathrm{d}\,z_{23\bar 1},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{23\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{23\bar 3} \right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{1\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{1\bar 1\bar 3},\ \mathrm{d}\,z_{1\bar 2\bar 3},\ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 2\bar 3} \right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{\bar 1\bar 2\bar 3} \right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar 1},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{123\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{123\bar 3} \right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 2\bar 3},\ \mathrm{d}\,z_{23\bar 2\bar 3},\ \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{23\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{23\bar 1\bar 3} \right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{1\bar 1\bar 2\bar 3}\ \ \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{2\bar 1\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{3\bar 1\bar 2\bar 3} \right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \mathrm{e}^{-\bar z_1}\,\mathrm{d}\,z_{123\bar 1\bar 2},\ \mathrm{e}^{\bar z_1}\,\mathrm{d}\,z_{123\bar 1\bar 3},\ \mathrm{d}\,z_{123\bar 2\bar 3} \right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar 1\bar 2\bar 3},\ \mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar 1\bar 2\bar 3},\ \mathrm{d}\,z_{23\bar 1\bar 2\bar 3} \right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \mathrm{d}\,z_{123\bar 1\bar 2\bar 3} \right\rangle$

TABLE 10. The double complex $C^{\bullet,\bullet}_\Gamma$ in (8) for the complex parallelizable Nakamura manifold in case *(b)*.

We compute $H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$ in case *(b)*, summarizing the results in Table 11.

The cochain complex $A^\bullet_\Gamma$ in (1) in case *(b)* is given in Table 12.

Finally, we summarize the results of the computations of the dimensions of the de Rham, the Dolbeault, and the Bott-Chern cohomologies in Table 13 (see [40, Example 2] for the Dolbeault cohomology).

Remark 2.28. Note that, for any $(p,q) \in \mathbb{Z}^2$,

$$\dim_\mathbb{C} H^{p,q}_{BC}(X) + \dim_\mathbb{C} H^{p,q}_{A}(X) \;=\; \dim_\mathbb{C} H^{p,q}_{\partial}(X) + \dim_\mathbb{C} H^{p,q}_{\bar\partial}(X)$$

in both case *(a)* and case *(b)*; note also that

$$\sum_{p+q=k} \left(\dim_\mathbb{C} H^{p,q}_{BC}(X) + \dim_\mathbb{C} H^{p,q}_{A}(X)\right) - 2\,\dim_\mathbb{C} H^{k}_{dR}(X;\mathbb{C}) \;=\; \begin{cases} 8 & \text{for } k \in \{1,\,5\} \\ 20 & \text{for } k \in \{2,\,4\} \\ 24 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case } (a)\;,$$

and

$$\sum_{p+q=k} \left(\dim_\mathbb{C} H^{p,q}_{BC}(X) + \dim_\mathbb{C} H^{p,q}_{A}(X)\right) - 2\,\dim_\mathbb{C} H^{k}_{dR}(X;\mathbb{C}) \;=\; \begin{cases} 4 & \text{for } k \in \{1,\,5\} \\ 8 & \text{for } k \in \{2,\,4\} \\ 8 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case } (b)\;.$$

case *(b)*	$H^{\bullet,\bullet}_{BC}(\Gamma\backslash G)$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\langle 1\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\langle[\mathrm{d}\,z_1]\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\langle[\mathrm{d}\,z_{\bar{1}}]\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\langle[\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13}],\ [\mathrm{d}\,z_{23}]\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\langle[\mathrm{d}\,z_{1\bar{1}}]\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\langle[\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{\bar{1}\bar{3}}],\ [\mathrm{d}\,z_{\bar{2}\bar{3}}]\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\langle[\mathrm{d}\,z_{123}]\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\langle[\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{1}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{1}}],\ [\mathrm{d}\,z_{23\bar{1}}]\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\langle[\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{1\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{1\bar{1}\bar{3}}],\ [\mathrm{d}\,z_{1\bar{2}\bar{3}}],\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\langle[\mathrm{d}\,z_{\bar{1}\bar{2}\bar{3}}]\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\langle[\mathrm{d}\,z_{123\bar{1}}]\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\langle[\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{2}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{2}\bar{3}}],\ [\mathrm{d}\,z_{23\bar{2}\bar{3}}],\ [\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{23\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{23\bar{1}\bar{3}}]\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\langle[\mathrm{d}\,z_{1\bar{1}\bar{2}\bar{3}}]\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\langle[\mathrm{e}^{-\bar{z}_1}\,\mathrm{d}\,z_{123\bar{1}\bar{2}}],\ [\mathrm{e}^{\bar{z}_1}\,\mathrm{d}\,z_{123\bar{1}\bar{3}}],\ [\mathrm{d}\,z_{123\bar{2}\bar{3}}]\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\langle[\mathrm{e}^{-z_1}\,\mathrm{d}\,z_{12\bar{1}\bar{2}\bar{3}}],\ [\mathrm{e}^{z_1}\,\mathrm{d}\,z_{13\bar{1}\bar{2}\bar{3}}],\ [\mathrm{d}\,z_{23\bar{1}\bar{2}\bar{3}}]\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\langle[\mathrm{d}\,z_{123\bar{1}\bar{2}\bar{3}}]\rangle$

Table 11. The Bott-Chern cohomology of the complex parallelizable Nakamura manifold in case *(b)*.

case *(b)*	$A^\bullet_\Gamma$
0	$\mathbb{C}\langle 1\rangle$
1	$\mathbb{C}\langle \mathrm{d}\,z_1,\ \mathrm{d}\,z_{\bar{1}}\rangle$
2	$\mathbb{C}\langle \mathrm{d}\,z_{1\bar{1}},\ \mathrm{d}\,z_{23},\ \mathrm{d}\,z_{\bar{2}\bar{3}}\rangle$
3	$\mathbb{C}\langle \mathrm{d}\,z_{123},\ \mathrm{d}\,z_{\bar{1}\bar{2}\bar{3}},\ \mathrm{d}\,z_{\bar{1}23},\ \mathrm{d}\,z_{1\bar{2}\bar{3}}\rangle$
4	$\mathbb{C}\langle \mathrm{d}\,z_{123\bar{1}},\ \mathrm{d}\,z_{23\bar{2}\bar{3}},\ \mathrm{d}\,z_{1\bar{1}\bar{2}\bar{3}}\rangle$
5	$\mathbb{C}\langle \mathrm{d}\,z_{23\bar{1}\bar{2}\bar{3}},\ \mathrm{d}\,z_{123\bar{2}\bar{3}}\rangle$
6	$\mathbb{C}\langle \mathrm{d}\,z_{123\bar{1}\bar{2}\bar{3}}\rangle$

Table 12. The cochain complex $A^\bullet_\Gamma$ in (1) for the complex parallelizable Nakamura manifold in case *(b)*.

2.5. **Currents.** Let X be a compact complex manifold, of complex dimension n. Denote the space of currents on X by $\mathrm{D}^{\bullet,\bullet}X := \mathrm{D}_{n-\bullet,n-\bullet}X$, namely, the topological dual space of $\wedge^{n-\bullet,n-\bullet}X$; endow $\mathrm{D}^{\bullet,\bullet}X$ with a structure of double complex, by defining $\partial\colon \mathrm{D}^{\bullet,\bullet}X \to \mathrm{D}^{\bullet+1,\bullet}X$ and $\overline{\partial}\colon \mathrm{D}^{\bullet,\bullet}X \to \mathrm{D}^{\bullet,\bullet+1}X$ by duality.

By means of the injective operator

$$T_\cdot\colon \wedge^{\bullet,\bullet} X \to \mathrm{D}^{\bullet,\bullet}X\ , \qquad T_\eta := \int_X \eta\wedge\cdot\ ,$$

which satisfies $T\circ\partial = \partial\circ T$ and $T\circ\overline{\partial} = \overline{\partial}\circ T$, consider the de Rham double complex $\left(\wedge^{\bullet,\bullet}X,\ \partial,\ \overline{\partial}\right)$ as a double sub-complex of $\left(\mathrm{D}^{\bullet,\bullet},\ \partial,\ \overline{\partial}\right)$.

For $(p,q)\in\mathbb{Z}^2$, denote the sheaf of p-holomorphic forms on X by Ω^p_X, denote the sheaf of (p,q)-forms on X by $\mathcal{A}^{p,q}_X$, and denote the sheaf of bi-degree (p,q)-currents by $\mathcal{D}^{p,q}_X$. Recall that, for any fixed $p\in\mathbb{Z}$, both

$$0\to\Omega^p_X\to\left(\mathcal{A}^{p,\bullet}_X,\ \overline{\partial}\right) \qquad \text{and} \qquad 0\to\Omega^p_X\to\left(\mathcal{D}^{p,\bullet}_X,\ \overline{\partial}\right)$$

$\dim_{\mathbb{C}} \mathbf{H}^{\bullet,\bullet}_{\sharp}(\mathbf{\Gamma}\backslash\mathbf{G})$	case *(a)*			case *(b)*		
	dR	$\overline{\partial}$	*BC*	*dR*	$\overline{\partial}$	*BC*
$(\mathbf{0},\mathbf{0})$	1	1	1	1	1	1
$(\mathbf{1},\mathbf{0})$	2	3	1	2	3	1
$(\mathbf{0},\mathbf{1})$		3	1		1	1
$(\mathbf{2},\mathbf{0})$	5	3	3	3	3	3
$(\mathbf{1},\mathbf{1})$		9	7		3	1
$(\mathbf{0},\mathbf{2})$		3	3		1	3
$(\mathbf{3},\mathbf{0})$	8	1	1	4	1	1
$(\mathbf{2},\mathbf{1})$		9	9		3	3
$(\mathbf{1},\mathbf{2})$		9	9		3	3
$(\mathbf{0},\mathbf{3})$		1	1		1	1
$(\mathbf{3},\mathbf{1})$	5	3	3	3	1	1
$(\mathbf{2},\mathbf{2})$		9	11		3	5
$(\mathbf{1},\mathbf{3})$		3	3		3	1
$(\mathbf{3},\mathbf{2})$	2	3	5	2	1	3
$(\mathbf{2},\mathbf{3})$		3	5		3	3
$(\mathbf{3},\mathbf{3})$	1	1	1	1	1	1

TABLE 13. Summary of the dimensions of the cohomologies of the complex parallelizable Nakamura manifold.

are fine (and hence acyclic, see, e.g., [30, IV.4.19]) resolutions of Ω^p_X, and hence

$$\frac{\ker\left(\overline{\partial}\colon\ \wedge^{p,\bullet} X \to \wedge^{p,\bullet+1}X\right)}{\operatorname{im}\left(\overline{\partial}\colon\ \wedge^{p,\bullet-1} X \to \wedge^{p,\bullet}X\right)} \simeq \check{H}^{\bullet}\left(X;\Omega^p_X\right) \simeq \frac{\ker\left(\overline{\partial}\colon \mathrm{D}^{p,\bullet}X \to \mathrm{D}^{p,\bullet+1}X\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathrm{D}^{p,\bullet-1}X \to \mathrm{D}^{p,\bullet}X\right)} ,$$

see, e.g., [30, IV.6.4].

Remark 2.29. More precisely, given X a compact complex manifold, for any $p \in \mathbb{Z}$ and for any $q \in \mathbb{Z}$, the maps $T_\cdot\colon (\wedge^{\bullet,q}X,\,\partial) \to (\mathrm{D}^{\bullet,q}X,\,\partial)$ and $T_\cdot\colon \left(\wedge^{p,\bullet}X,\,\overline{\partial}\right) \to \left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$ are quasi-isomorphisms.

Indeed, firstly, we show that $T_\cdot\colon \left(\wedge^{p,\bullet}X,\,\overline{\partial}\right) \to \left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$ induces an injective map in cohomology. Fix g a Hermitian metric on X. If $T_{[\alpha]} = \left[\overline{\partial}S\right] = [0] \in H^{\bullet}\left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$ with α the $\overline{\square}_g$-harmonic representative of $[\alpha] \in H^{\bullet}\left(\wedge^{p,\bullet}X,\,\overline{\partial}\right)$ and $S \in \mathrm{D}^{p,\bullet-1}X$, then in particular $T_\alpha\lfloor_{\ker\overline{\partial}} = 0$. Since $\bar{*}_g\alpha \in \ker\overline{\partial}$, it follows that $0 = T_\alpha\left(\bar{*}_g\alpha\right) = \int_X \alpha\wedge\bar{*}_g\alpha$ and hence $\alpha = 0$. Now, since $\frac{\ker\left(\overline{\partial}\colon\ \wedge^{p,\bullet}X\to\wedge^{p,\bullet+1}X\right)}{\operatorname{im}\left(\overline{\partial}\colon\ \wedge^{p,\bullet-1}X\to\wedge^{p,\bullet}X\right)}$ and $\frac{\ker\left(\overline{\partial}\colon\ \mathrm{D}^{p,\bullet}X\to\mathrm{D}^{p,\bullet+1}X\right)}{\operatorname{im}\left(\overline{\partial}\colon\ \mathrm{D}^{p,\bullet-1}X\to\ correnti^{p,\bullet}X\right)}$ are isomorphic $\mathbb{C}$-vector spaces of finite dimension, it follows that $T_\cdot\colon \left(\wedge^{p,\bullet}X,\,\overline{\partial}\right) \to \left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$ is actually a quasi-isomorphism. By conjugation, also $T_\cdot\colon (\wedge^{\bullet,q}X,\,\partial) \to (\mathrm{D}^{\bullet,q}X,\,\partial)$ is a quasi-isomorphism.

By applying Proposition 1.1 to $\left(\wedge^{p,\bullet}X,\,\overline{\partial}\right) \hookrightarrow \left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$, or by noting that both $0 \to \underline{\mathbb{C}}_X \to (\mathcal{A}^{\bullet}_X\otimes\mathbb{C},\,\mathrm{d})$ and $0 \to \underline{\mathbb{C}}_X \to (\mathcal{D}^{\bullet}_X\otimes\mathbb{C},\,\mathrm{d})$ are acyclic resolutions of the constant sheaf $\underline{\mathbb{C}}_X$ over X (where, for $k \in \mathbb{Z}$, the sheaf of k-forms on X is denoted by $\mathcal{A}^k_X$, and the sheaf of degree k-currents is denoted by $\mathcal{D}^k_X$), one gets that

$$\frac{\ker\left(\mathrm{d}\colon\ \wedge^{\bullet} X\otimes_{\mathbb{R}}\mathbb{C} \to \wedge^{\bullet+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)}{\operatorname{im}\left(\overline{\partial}\colon\ \wedge^{\bullet-1} X\otimes_{\mathbb{R}}\mathbb{C} \to \wedge^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \simeq \check{H}^{\bullet}\left(X;\underline{\mathbb{C}}_X\right) \simeq \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{\bullet+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)}{\operatorname{im}\left(\mathrm{d}\colon \mathrm{D}^{\bullet-1}X\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C}\right)} .$$

Lemma 2.30. *Let X be a compact complex manifold. For any $(p,q) \in \mathbb{Z}^2$, the map $T_\cdot\colon \wedge^{\bullet,\bullet}X \to \mathrm{D}^{\bullet,\bullet}X$ induces the isomorphism*

$$T_\cdot\colon \frac{\ker\left(\mathrm{d}\colon\ \wedge^{p,q} X \to \wedge^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)}{\operatorname{im}\left(\mathrm{d}\colon\ \wedge^{p+q-1} X\otimes_{\mathbb{R}}\mathbb{C} \to \wedge^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} \to \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{p,q}X \to \mathrm{D}^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)}{\operatorname{im}\left(\mathrm{d}\colon \mathrm{D}^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)} .$$

Proof. Consider the regularization process in [31, Theorem III.12]: there exist $R\colon \mathrm{D}^{\bullet,\bullet}X \to \wedge^{\bullet,\bullet}X$ and $A\colon \mathrm{D}^{\bullet}X\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{\bullet+1}X\otimes_{\mathbb{R}}\mathbb{C}$ linear operators such that

$$\mathrm{id}_{\mathrm{D}^{\bullet,\bullet}X} \;=\; R + \mathrm{d}\,A + A\,\mathrm{d}\;, \qquad \text{and} \qquad R\lfloor_{\wedge^{\bullet,\bullet}X} \;=\; \mathrm{id}_{\wedge^{\bullet,\bullet}X} \text{ and } A\lfloor_{\wedge^{\bullet,\bullet}X} \;=\; 0\;.$$

Take $S \in \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{p,q}X\to\mathrm{D}^{p+q+1}X\otimes_{\mathbb{R}}\mathbb{C}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{D}^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}\to\mathrm{D}^{p+q}X\otimes_{\mathbb{R}}\mathbb{C}\right)}$. Since the map $T_{\cdot}\colon \wedge^{\bullet,\bullet}X \to \mathrm{D}^{\bullet,\bullet}X$ is a quasi-isomorphism, then there exist $\eta\in\ker\mathrm{d}\cap\wedge^{p,q}X$ and $U\in\mathrm{D}^{p+q-1}X\otimes_{\mathbb{R}}\mathbb{C}$ such that

$$S \;=\; T_\eta + \mathrm{d}\,U\;;$$

hence one gets

$$RS \;=\; T_\eta + \mathrm{d}\left(U - AS\right)\;,$$

and hence the lemma follows. □

As a consequence, by using Theorem 1.3, we get another proof of the following result by M. Schweitzer: see [64], and also [47, §3.4], where it is noticed as a consequence of the hypercohomological interpretation of the Bott-Chern cohomology, see also [30, IV.12.1].

Corollary 2.31 (see [64, §4.d])**.** *Let X be a compact complex manifold. Then, for any $(p,q)\in\mathbb{Z}^2$, the natural map*

$$T_{\cdot}\colon \frac{\ker\left(\partial+\overline{\partial}\colon\; \wedge^{p,q}X\to\wedge^{p+1,q}X\oplus\wedge^{p,q+1}X\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon\; \wedge^{p-1,q-1}X\to\wedge^{p,q}X\right)} \to \frac{\ker\left(\partial+\overline{\partial}\colon \mathrm{D}^{p,q}X\to\mathrm{D}^{p+1,q}X\oplus\mathrm{D}^{p,q+1}X\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathrm{D}^{p-1,q-1}X\to\mathrm{D}^{p,q}X\right)}$$

induced by $T_{\cdot}\colon \wedge^{\bullet,\bullet}X\ni\eta\mapsto T_\eta:=\int_X\eta\wedge\cdot\in\mathrm{D}^{\bullet,\bullet}X$ is an isomorphism.

Proof. We firstly prove that $T_{\cdot}$ induces an injective map in Bott-Chern cohomology. Indeed, let $\mathfrak{a}=[\alpha]\in H^{p,q}_{BC}(X)$ be such that $[T_{\mathfrak{a}}]=0\in\frac{\ker\left(\partial+\overline{\partial}\colon \mathrm{D}^{p,q}X\to\mathrm{D}^{p+1,q}X\oplus\mathrm{D}^{p,q+1}X\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathrm{D}^{p-1,q-1}X\to\mathrm{D}^{p,q}X\right)}$. Choose g a Hermitian metric on X, and let $\alpha\in\wedge^{p,q}X$ be the $\tilde{\Delta}^{BC}$-harmonic representative of $\mathfrak{a}$ with respect to g. Therefore, there exists $S\in\mathrm{D}^{p-1,q-1}X$ such that $T_\alpha=\partial\overline{\partial}S$. In particular, $T_\alpha\lfloor_{\ker\partial\overline{\partial}}=0$. Since $\bar{*}_g\alpha\in\ker\partial\overline{\partial}$, it follows that $0=T_\alpha\left(\bar{*}_g\alpha\right)=\int_X\alpha\wedge\bar{*}_g\alpha$, and hence $\mathfrak{a}=[\alpha]=0$.

We prove now that $T_{\cdot}$ induces a surjective map in Bott-Chern cohomology. Firstly, by Remark 2.29, for any $p\in\mathbb{Z}$ and for any $q\in\mathbb{Z}$, the maps $T_{\cdot}\colon \left(\wedge^{\bullet,q}X,\,\partial\right)\to\left(\mathrm{D}^{\bullet,q}X,\,\partial\right)$ and $T_{\cdot}\colon \left(\wedge^{p,\bullet}X,\,\overline{\partial}\right)\to\left(\mathrm{D}^{p,\bullet}X,\,\overline{\partial}\right)$ are quasi-isomorphisms. Furthermore, by Lemma 2.30, the induced map

$$T_{\cdot}\colon \frac{\ker\left(\mathrm{d}\colon\; \wedge^{\bullet}X\otimes\mathbb{C}\to\wedge^{\bullet+1}X\otimes\mathbb{C}\right)\cap\wedge^{p,q}X}{\mathrm{im}\left(\mathrm{d}\colon\; \wedge^{\bullet-1}X\otimes\mathbb{C}\to\wedge^{\bullet}X\otimes\mathbb{C}\right)} \to \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{\bullet}X\otimes\mathbb{C}\to\mathrm{D}^{\bullet+1}X\otimes\mathbb{C}\right)\cap\mathrm{D}^{p,q}X}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{D}^{\bullet-1}X\otimes\mathbb{C}\to\mathrm{D}^{\bullet}X\otimes\mathbb{C}\right)}$$

is surjective. Hence, Theorem 1.3 applies, yielding that the map $T_{\cdot}$ induces a surjective map in Bott-Chern cohomology. □

Remark 2.32. Given X a compact complex manifold of complex dimension n and G a finite group of biholomorphisms of X, consider the compact complex *orbifold* $\tilde{X}:=X/G$ of complex dimension n (namely, [63, Definition 2], $\tilde{X}$ is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$ with $G\subset\mathrm{GL}\left(\mathbb{C}^n\right)$ finite; see [18, Theorem 1], see also [57, Theorem 1.7.2]).

By extending the action of G on X to $\wedge^{\bullet}X$, respectively $\wedge^{\bullet,\bullet}X$, set $\wedge^{\bullet}\tilde{X}$ the space of G-invariant forms in $\wedge^{\bullet}X$, respectively $\wedge^{\bullet,\bullet}\tilde{X}$ the space of G-invariant forms in $\wedge^{\bullet,\bullet}X$. Analogously, consider $\mathrm{D}^{\bullet}\tilde{X}$ the space of G-invariant currents in $\mathrm{D}^{\bullet}X$, respectively $\mathrm{D}^{\bullet,\bullet}\tilde{X}$ the space of G-invariant currents in $\mathrm{D}^{\bullet,\bullet}X$.

Consider the sub-complex $T_{\cdot}\colon \left(\wedge^{\bullet,\bullet}\tilde{X},\,\partial,\,\overline{\partial}\right)\hookrightarrow\left(\mathrm{D}^{\bullet,\bullet}\tilde{X},\,\partial,\,\overline{\partial}\right)$. By W. L. Baily's result [12, page 807], and arguing as in Remark 1.9 by means of a Hermitian metric on $\tilde{X}$, namely, a G-invariant Hermitian metric on X, it follows that, for any $p\in\mathbb{Z}$, the induced inclusion $T_{\cdot}\colon \left(\wedge^{p,\bullet}\tilde{X},\,\overline{\partial}\right)\hookrightarrow\left(\mathrm{D}^{p,\bullet}\tilde{X},\,\overline{\partial}\right)$ is a quasi-isomorphism; by conjugation, it follows also that, for any $q\in\mathbb{Z}$, the induced inclusion $T_{\cdot}\colon \left(\wedge^{\bullet,q}\tilde{X},\,\partial\right)\hookrightarrow\left(\mathrm{D}^{\bullet,q}\tilde{X},\,\partial\right)$ is a quasi-isomorphism. In particular, by using Proposition 1.1, one recovers that the induced inclusion $T_{\cdot}\colon \left(\wedge^{\bullet}\tilde{X},\,\mathrm{d}\right)\hookrightarrow\left(\mathrm{D}^{\bullet}\tilde{X},\,\mathrm{d}\right)$ is a quasi-isomorphism, as proved also by I. Satake, [63, Theorem 1].

We note that the inclusion $T_\cdot\colon\ \wedge^{\bullet,\bullet}\tilde{X} \to \mathrm{D}^{\bullet,\bullet}\tilde{X}$ induces the surjective map

$$T_\cdot\colon \frac{\ker\left(\mathrm{d}\colon\ \wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C} \to \wedge^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\wedge^{p,q}\tilde{X}}{\mathrm{im}\left(\mathrm{d}\colon\ \wedge^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C} \to \wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)}$$

$$\to \quad \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\cap\mathrm{D}^{p,q}\tilde{X}}{\mathrm{im}\left(\mathrm{d}\colon \mathrm{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)}\ ;$$

indeed, since $g^*\circ T\circ g^* = T$ for any $g\in G$, the regularization (see [31, Theorem III.12]) of a G-invariant current of bidegree (p,q) gives a G-invariant (p,q)-form.

Hence, Theorem 1.3 applies, yielding that, for any $(p,q)\in\mathbb{Z}^2$, the inclusion $T_\cdot$ induces an isomorphism

$$T_\cdot\colon \frac{\ker\left(\mathrm{d}\colon\ \wedge^{p,q}\tilde{X} \to \wedge^{p+1,q}\tilde{X}\oplus\wedge^{p,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon\ \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)} \stackrel{\simeq}{\to} \frac{\ker\left(\mathrm{d}\colon \mathrm{D}^{p,q}\tilde{X} \to \mathrm{D}^{p+1,q}\tilde{X}\oplus\mathrm{D}^{p,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathrm{D}^{p-1,q-1}\tilde{X} \to \mathrm{D}^{p,q}\tilde{X}\right)}\ ,$$

as proved also in [5, Theorem 1].

Note that one can argue also by means of the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [30, §V I.12.1] and M. Schweitzer, [64, §4], see also [47, §3.2].

Remark 2.33 ([8])**.** We note that the results in Section 1 can be used also to investigate the symplectic Bott-Chern and Aeppli cohomologies, as introduced and studied by L.-S. Tseng and S.-T. Yau in [66, 67, 68], for solvmanifolds endowed with left-invariant symplectic structures. In particular, one gets a different proof of the result in [50, Theorem 3] by M. Macrì for completely-solvable solvmanifolds, and a generalization for (non-necessarily completely-solvable) solvmanifolds. The complex parallelizable Nakamura manifold $\Gamma\backslash G$ can be investigated explicitly, also in relation with the validity of the $\mathrm{d}\,\mathrm{d}^{\Lambda}$-lemma, equivalently, the Hard Lefschetz Condition; see also [38]. We refer to [8] for more details.

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