ON COHOMOLOGICAL DECOMPOSABILITY OF ALMOST–KÄHLER STRUCTURES

DANIELE ANGELLA, ADRIANO TOMASSINI, AND WEIYI ZHANG

Abstract. We study the J-invariant and J-anti-invariant cohomological subgroups of the de Rham cohomology of a compact manifold M endowed with an almost-Kähler structure $(J,\, \omega,\, g)$. In particular, almost-Kähler manifolds satisfying a Lefschetz type property, and solvmanifolds endowed with left-invariant almost-complex structures are investigated.

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Introduction

Cohomological properties of compact complex, and, more in general, almost-complex, manifolds have been recently studied by many authors, see, e.g., [3], respectively [11, 12], and the references therein. The study of the cohomology of almost-complex manifolds is motivated, in particular, by a question of Donaldson's, [10, Question 2], relating the tamed and compatible symplectic cones of a compact 4-dimensional almost-complex manifold, see, e.g., [20], and by the analogous question arising for compact higher dimensional complex manifolds, see [20, page 678] and [26, Question 1.7]. (We recall that a symplectic structure ω on a manifold M is said to *tame* an almost-complex structure J if $\omega_x\,(u_x,\, J_x u_x) > 0$ for any $x \in M$ and for any $u \in T_x M \setminus \{0\}$, and it is said *compatible* with J if $g := \omega(\cdot,\, J\,\cdot\cdot)$ is a J-Hermitian metric; in the latter case, the triple $(J,\, \omega,\, g)$ is called an *almost-Kähler structure* on M.)

Following T.-J. Li and the third author, [20], an almost-complex structure J on a $2n$-dimensional manifold M is called $\mathcal{C}^\infty$*-pure-and-full* if

$$H^2_{dR}(M;\mathbb{R}) = H^{(1,1)}_J(M)_\mathbb{R} \oplus H^{(2,0),(0,2)}_J(M)_\mathbb{R}\,,$$

where $H^{(1,1)}_J(M)_\mathbb{R}$ and $H^{(2,0),(0,2)}_J(M)_\mathbb{R}$ denote the subgroups of $H^2_{dR}(M;\mathbb{R})$ whose elements can be represented by forms of type $(1,1)$ and $(2,0)+(0,2)$ respectively. In the notation of T. Drăghici, T.-J. Li, and the third author, [11], $H^{(1,1)}_J(M)_\mathbb{R} =: H^+_J(M)$ and $H^{(2,0),(0,2)}_J(M)_\mathbb{R} =: H^-_J(M)$ are the J*-invariant* and the J*-anti-invariant cohomology subgroups* respectively.

In [11, Theorem 2.3], T. Drăghici, T.-J. Li, and the third author proved that every almost-complex structure on a compact 4-dimensional manifold is $\mathcal{C}^\infty$-pure-and-full. This is no more true in dimension greater than four, see, e.g., [15, Example 3.3], see also [1, 2].

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The groups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ appear as a natural generalization of the Dolbeault cohomology groups to the non-integrable case, see, e.g., [20, Proposition 2.1]. In fact, compact Kähler manifolds are $\mathcal{C}^\infty$-pure-and-full, and, in this case, $H_J^{(1,1)}(M)_{\mathbb{R}} \simeq H^{1,1}_{\overline{\partial}}(M) \cap H^2_{dR}(M;\mathbb{R})$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \simeq \left(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M)\right) \cap H^2_{dR}(M;\mathbb{R})$.

We remark that, on a compact complex manifold, other cohomologies can be defined, namely, the Bott-Chern and Aeppli cohomologies. In [3], the problem of cohomology decomposition in terms of the Bott-Chern cohomology groups is investigated, providing in particular a characterization of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma.

Compact Kähler manifolds being $\mathcal{C}^\infty$-pure-and-full, in this paper we are interested in the study of the cohomological subgroups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ for almost-Kähler manifolds.

On the one hand, A. Fino and the second author, [15, Proposition 3.2], as well as T. Drăghici, T.-J. Li, and the third author, [11, Proposition 2.8], proved that the almost-complex structure of a compact almost-Kähler manifold is $\mathcal{C}^\infty$-pure. On the other hand, we prove the following result, showing therefore a difference between the integrable and the non-integrable cases.

Proposition 4.1. *Let* $X := \mathbb{Z}[\mathrm{i}]^3 \Big\backslash \left(\mathbb{C}^3, *\right)$ *be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure* $(J,\,\omega,\,g)$ *on* X *which is* $\mathcal{C}^\infty$*-pure and non-*$\mathcal{C}^\infty$*-full. Furthermore, the* Lefschetz type operator $\mathcal{L}_\omega := \omega \wedge \cdot\colon \wedge^2 M \to \wedge^4 M$ *of the almost-Kähler structure* $(J,\,\omega,\,g)$ *does not take* g*-harmonic* 2*-forms to* g*-harmonic* 4*-forms.*

In studying cohomological decomposition of the de Rham cohomology of almost-Kähler manifolds, the third author introduced a *Lefschetz type property* for 2-forms, see Definition 2.2. Such a property is stronger than the Hard Lefschetz Condition on 2-classes, namely, the property that $[\omega]^{n-2} \smile \cdot\colon H^2_{dR}(M;\mathbb{R}) \to H^{2n-2}_{dR}(M;\mathbb{R})$ is an isomorphism, where $2n := \dim M$.

We study such a Lefschetz type property on almost-Kähler manifolds $(M,\ J,\ \omega,\ g)$ in relation to the existence of a cohomological decomposition of $H^2_{dR}(M;\mathbb{R})$. More precisely, we prove the following result.

Theorem 2.3. *Let* $(M,\,J,\,\omega,\,g)$ *be a compact almost-Kähler manifold. Suppose that there exists a basis of* $H^2_{dR}(X;\mathbb{R})$ *represented by* g*-harmonic* 2*-forms which are of pure type with respect to* J*. Then the Lefschetz type property on* 2*-forms is satisfied.*

Note that, by the hypothesis, it follows,in particular, that J is $\mathcal{C}^\infty$-pure-and-full and pure-and-full, [15, Theorem 3.7]. Note also that A. Fino and the second author provided in [15] several examples of compact non-Kähler solvmanifolds admitting a basis of harmonic representatives of pure-type with respect to the almost-complex structure. In [13, §2], T. Drăghici, T.-J. Li, and the third author ask whether such a Lefschetz type property on 2-forms is actually equivalent to $\mathcal{C}^\infty$-fullness for every almost-Kähler nilmanifold and solvmanifold, without any further assumption; Theorem 2.3 and Proposition 4.1 provide results and examples in favour of a possibly positive answer to their question.

In [12, Theorem 1.1], starting with a compact complex surface $(M,\,J)$, it is shown that the dimension $h^-_{\tilde{J}}$ of the $\tilde{J}$-anti-invariant cohomology subgroup $H^-_{\tilde{J}}(M)$

of any *metric related* almost-complex structure $\tilde{J}$ on M (namely, an almost-complex structure $\tilde{J}$ on M inducing the same orientation as that one induced by J and with a common compatible metric), such that $\tilde{J} \neq \pm J$, can be 0, 1, or 2, and a description of such almost-complex structures $\tilde{J}$ having $h^-_{\tilde{J}} \in \{1, 2\}$ is provided. Furthermore, it is conjectured that $h^-_J = 0$ for a generic almost-complex structure J on a compact 4-dimensional manifold, and that if $h^-_J \geq 3$, then J is integrable, [12, Conjecture 2.4, Conjecture 2.5]. One could set a similar question for higher dimensional manifolds, asking Question 5.2: *are there examples of non-integrable almost-complex structures J on a compact $2n$-dimensional manifold with $h^-_J > n\,(n-1)$?*.

Finally, we prove a Nomizu-type result for the subgroups $H^{\pm}_J(M)$ of a completely-solvable solvmanifolds $M = \Gamma\backslash G$ endowed with left-invariant almost-complex structures J. More precisely, denote the Lie algebra associated to G by $\mathfrak{g}$, and consider

$$H^{(p,q),(q,p)}_J(\mathfrak{g})_{\mathbb{R}} \;:=\; \left\{\mathfrak{a} = [\alpha] \in H^\bullet\left(\wedge^\bullet \mathfrak{g}^*,\, \mathrm{d}\right) \;:\; \alpha \in \wedge^{(p,q),(q,p)}_J \mathfrak{g}^*\right\} \;\subseteq\; H^\bullet_{dR}(M;\mathbb{R})$$

the subgroup of $H^\bullet_{dR}(M;\mathbb{R})$ that consists of classes admitting a left-invariant representative of type $(p,q)+(q,p)$, where $\wedge^{(p,q),(q,p)}_J \mathfrak{g}^* := \left(\wedge^{p,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \oplus \wedge^{q,p}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right) \cap \wedge^\bullet \mathfrak{g}^*$; then the following result holds.

Theorem 5.4. *Let $M = \Gamma\backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. For any $p, q \in \mathbb{N}$, the map $j\colon H^{(p,q),(q,p)}_J(\mathfrak{g})_{\mathbb{R}} \to H^{(p,q),(q,p)}_J(M)_{\mathbb{R}}$ induced by left-translations is injective, and, if $H^\bullet_{dR}\left(\wedge^\bullet \mathfrak{g}^*,\, \mathrm{d}\right) \simeq H^\bullet_{dR}(M;\mathbb{R})$ (for instance, if M is a completely-solvable solvmanifold), then $j\colon H^{(p,q),(q,p)}_J(\mathfrak{g})_{\mathbb{R}} \to H^{(p,q),(q,p)}_J(M)_{\mathbb{R}}$ is in fact an isomorphism.*

In particular, it follows that $\dim_{\mathbb{R}} H^-_J(M) \leq n\,(n-1)$ for every left-invariant almost-complex structure on a completely-solvable solvmanifold.

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1. $\mathcal{C}^\infty$-pure-and-full almost-complex structures

1.1. **Subgroups of the de Rham cohomology of an almost-complex manifolds.** We start by fixing some notation and recalling some recent results on cohomological properties of almost-complex manifolds; for more details see, e.g., [20, 11, 12, 15, 1, 2, 13], and the references therein.

Let J be a smooth almost-complex structure on a compact $2n$-dimensional manifold M. Denote by $\wedge^r M$ the bundle of r-forms on M; we denote with the same symbol $\wedge^r M := \Gamma(M, \wedge^r M)$ the space of smooth global sections of the bundle $\wedge^r M$. Then J extends to a complex automorphism of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ such that $T^{\mathbb{C}}M = T^{1,0}_J M \oplus T^{0,1}_J M$, where $T^{1,0}_J M$ and $T^{0,1}_J M$ are the $(\pm\,\mathrm{i})$-eigenbundles. The action of J can be extended to the space $\wedge^r(M;\mathbb{C})$ of smooth global sections of the bundle $\wedge^r(M;\mathbb{C}) := \wedge^r M \otimes \mathbb{C}$ getting the following decomposition:

$$\wedge^r(M;\mathbb{C}) = \bigoplus_{p+q=r} \wedge_J^{p,q} M \; .$$

Then the space $\wedge^r M$ of real smooth differential r-forms decomposes as

$$\wedge^r M = \bigoplus_{p+q=r,\; p\leq q} \wedge_J^{(p,q),(q,p)} (M)_{\mathbb{R}} \; ,$$

where, for $p<q$, (later on, we do not distinguish the cases $p<q$ and $p=q$,)

$$\wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} := \left\{\alpha \in \wedge_J^{p,q} M \oplus \wedge_J^{q,p} M \;:\; \alpha = \overline{\alpha}\right\} \;, \quad \wedge_J^{(p,p)}(M)_{\mathbb{R}} := \left\{\alpha \in \wedge_J^{p,p} M \;:\; \alpha = \overline{\alpha}\right\}$$

In particular, for $r=2$, we will adopt the following notation:

$$\wedge_J^{1,1}(M)_{\mathbb{R}} =: \wedge_J^{+} M \;, \quad \wedge_J^{(2,0),(0,2)}(M)_{\mathbb{R}} =: \wedge_J^{-} M \;;$$

this is consistent with the decomposition in invariant and anti-invariant part of $\wedge^2 M$ under the natural action of J on $\wedge^2 M$, given by $J\alpha(\cdot,\,\cdot) := \alpha(J\cdot,\, J\cdot)$.
We will refer to forms in $\wedge_J^{1,1}(M)_{\mathbb{R}}$, respectively $\wedge_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ as forms of *pure type with respect to* J.

For a finite set S of pairs of integers, let

$$\mathcal{Z}_J^S := \bigoplus_{(p,q)\in S,\; p\leq q} \mathcal{Z}_J^{(p,q),(q,p)} \;, \qquad \mathcal{B}_J^S := \bigoplus_{(p,q)\in S,\; p\leq q} \mathcal{B}_J^{(p,q),(q,p)} \;,$$

where

$$\mathcal{Z}_J^{(p,q),(q,p)} := \left\{\alpha \in \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \;:\; \mathrm{d}\,\alpha = 0\right\} \;,$$

$$\mathcal{B}_J^{(p,q),(q,p)} := \left\{\beta \in \wedge_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \;:\; \text{there exists } \gamma \text{ such that } \mathrm{d}\,\gamma = \beta\right\} \; .$$

Define

$$H_J^S(M)_{\mathbb{R}} := \frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} \; .$$

Let $\mathcal{B}$ be the space of d-exact forms. Since $\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} = \frac{\mathcal{Z}_J^S}{\mathcal{B}\cap\mathcal{Z}_J^S}$, a natural inclusion $\rho_S \colon \frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} \to \frac{\mathcal{Z}_J^S}{\mathcal{B}}$ is defined. As in [20], we will write $\rho_S\left(\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S}\right)$ simply as $\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S}$ and consequently the cohomology spaces $H_J^S(M)_{\mathbb{R}}$ can be identified as

$$H_J^S(M)_{\mathbb{R}} = \left\{[\alpha] \in H_{dR}^{\bullet}(M;\mathbb{R}) \;:\; \alpha \in \mathcal{Z}_J^S\right\} = \frac{\mathcal{Z}_J^S}{\mathcal{B}} \; .$$

Therefore, there is a natural inclusion

$$H_J^{(1,1)}(M)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \subseteq H_{dR}^2(M;\mathbb{R}) \; .$$

1.2. $\mathcal{C}^\infty$**-pure-and-full and pure-and-full almost-complex structures.** As in [20], we set the following definition.

Definition 1.1 ([20, Definition 2.2, Definition 2.3, Lemma 2.2])**.** An almost-complex structure J on a manifold M is said to be

- $\mathcal{C}^\infty$*-pure* if $H_J^{(1,1)}(M)_{\mathbb{R}} \cap H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\}$,
- $\mathcal{C}^\infty$*-full* if $H_{dR}^2(M;\mathbb{R}) = H_J^{(1,1)}(M)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$,
- $\mathcal{C}^\infty$*-pure-and-full* if

$$H_{dR}^2(M;\mathbb{R}) \;=\; H_J^{(1,1)}(M)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \; .$$

According to the previous notation, we will write

$$H_J^+(M) \;:=\; H_J^{(1,1)}(M)_{\mathbb{R}}\;, \qquad H_J^-(M) \;:=\; H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}\;.$$

Similar definitions in terms of currents can be given, introducing the notion of *pure-and-full* almost-complex structure: we refer to [20, §2.2.2] for further details and results. More precisely, on an almost complex manifold $(M,\, J)$, the space $\mathcal{E}_k(M)_{\mathbb{R}}$ of real k-currents has a decomposition $\mathcal{E}_k(M)_{\mathbb{R}} = \bigoplus_{\substack{p+q=k\\ p\leq q}} \mathcal{E}^J_{(p,q),(q,p)}(M)_{\mathbb{R}}$, where $\mathcal{E}^J_{(p,q),(q,p)}(M)_{\mathbb{R}}$ denotes the space of real k-currents of bi-dimension $(p,q)+(q,p)$.

Let $\mathcal{Z}^J_{(2,0),(0,2)}$ and $\mathcal{Z}^J_{(1,1)}$ denote the spaces of real d-closed currents of bi-dimension $(2,0)+(0,2)$, respectively $(1,1)$, and $\mathcal{B}^J_{(2,0),(0,2)}$ and $\mathcal{B}^J_{(1,1)}$ denote the spaces of real d-exact currents of bi-dimension $(2,0)+(0,2)$, respectively $(1,1)$. Denote by $\mathcal{B}$ the space of boundaries. Let, as in [20],

$$H^J_{(1,1)}(M)_{\mathbb{R}} \quad := \quad \left\{[\alpha]\in H_2(M;\mathbb{R}) \;:\; \alpha\in \mathcal{Z}^J_{(1,1)}\right\} \;=\; \frac{\mathcal{Z}^J_{(1,1)}}{\mathcal{B}}\;,$$

$$H^J_{(2,0),(0,2)}(M)_{\mathbb{R}} \quad := \quad \left\{[\alpha]\in H_2(M;\mathbb{R}) \;:\; \alpha\in \mathcal{Z}^J_{(2,0),(0,2)}\right\} \;=\; \frac{\mathcal{Z}^J_{(2,0),(0,2)}}{\mathcal{B}}\;.$$

We recall the following definition.

Definition 1.2 ([20, Definition 2.15, Definition 2.16])**.** An almost complex structure J on a manifold M is said to be *pure* if $H^J_{(1,1)}(M)_{\mathbb{R}} \cap H^J_{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\}$. It is said to be *full* if $H_2(M;\mathbb{R}) = H^J_{(1,1)}(M)_{\mathbb{R}} + H^J_{(2,0),(0,2)}(M)_{\mathbb{R}}$. Therefore, an almost complex structure J is *pure-and-full* if and only if

$$H_2(M,\mathbb{R}) \;=\; H^J_{(1,1)}(M)_{\mathbb{R}} \oplus H^J_{(2,0),(0,2)}(M)_{\mathbb{R}}\;.$$

In [20, Proposition 2.1] it is shown that, given a compact complex manifold $(M,\, J)$ of complex dimension n, if $n=2$ or J is Kähler, then J is $\mathcal{C}^\infty$-pure-and-full, and $H_J^{(1,1)}(M)_{\mathbb{R}} \simeq H^{1,1}_{\overline{\partial}}(M) \cap H^2_{dR}(M;\mathbb{R})$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \simeq \left(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M)\right) \cap H^2_{dR}(M;\mathbb{R})$. In view of this result, the subgroups $H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$ of the de Rham cohomology can be viewed as an analogue of the Dolbeault cohomology groups for non-integrable almost-complex structures.

In [11, Theorem 2.3] it is proven the following result.

Theorem 1.3 ([11, Theorem 2.3])**.** *If M is a compact manifold of dimension 4, then any almost-complex structure J on M is $\mathcal{C}^\infty$-pure-and-full.*

This is no more true in dimension higher than 4: in [15, Example 3.3], a compact non-$\mathcal{C}^\infty$-pure almost-complex structure on a 6-dimensional nilmanifold is constructed. Therefore, the previous theorem can be considered a sort of Hodge decomposition theorem in the non-Kähler case.

2. Cohomological properties of almost-Kähler manifolds

2.1. **Lefschetz type property on almost-Kähler manifolds with pure-type harmonic representatives.** Given a compact $2n$-dimensional almost-Kähler manifold $(M,\, J,\, \omega,\, g)$, we are interested in studying the property of being $\mathcal{C}^\infty$-pure-and-full.

Firstly we recall the following result.

Proposition 2.1 ([11, Proposition 2.8], [15, Proposition 3.2]). *If J is an almost-complex structure on a compact manifold M and J admits a compatible symplectic structure, then J is $\mathcal{C}^\infty$-pure.*

Furthermore, A. Fino and the second author proved that an almost-Kähler manifold admitting a basis of harmonic 2-forms whose elements are of pure type with respect to the almost-complex structure is $\mathcal{C}^\infty$-pure-and-full and pure-and-full, [15, Theorem 3.7]; they also provided several examples of compact non-Kähler solvmanifolds satisfying the above assumption in [15].

To the purpose of studying the property of being $\mathcal{C}^\infty$-pure-and-full on almost-Kähler manifolds, we recall the following definition.

Definition 2.2. Given a compact $2n$-dimensional symplectic manifold (M, ω), denote by

$$\mathcal{L}_\omega \colon \wedge^2 M \to \wedge^{2n-2} M\,, \qquad \mathcal{L}_\omega(\alpha) := \omega^{n-2} \wedge \alpha\,,$$

the *Lefschetz type operator* (on 2-forms) associated with ω.
Then one says that the compact $2n$-dimensional almost-Kähler manifold $(M,\, J,\, \omega,\, g)$ satisfies the *Lefschetz type property (on 2-forms)* if $\mathcal{L}_\omega$ takes g-harmonic 2-forms to g-harmonic $(2n-2)$-forms.

Furthermore, we recall some notions and results from [6, 22, 27], see also [23, 7]. Let $(M,\, \omega)$ be a compact $2n$-dimensional symplectic manifold. Extend $\omega^{-1} \colon T^*M \to TM$ to the whole exterior algebra of T^*M. For any $k \in \mathbb{N}$, the *symplectic $\star_\omega$ operator* is defined as

$$\star_\omega \colon \wedge^k M \to \wedge^{2n-k} M\,, \qquad \beta \wedge \star_\omega \alpha \;=\; \omega^{-1}(\alpha, \beta)\, \frac{\omega^n}{n!}\,, \quad \forall \alpha,\, \beta \in \wedge^k M\,.$$

One can prove that $\star_\omega^2 = \mathrm{id}_{\wedge^\bullet M}$, [6, Lemma 2.1.2].
For any $k \in \mathbb{N}$, define the *symplectic co-differential operator*

$$\delta_\omega \colon \wedge^k M \to \wedge^{k-1} M\,, \qquad \delta_\omega\lfloor_{\wedge^k M} :=\; (-1)^{k+1}\, \star_\omega\, \mathrm{d} \star_\omega\,;$$

this operator has been studied by J.-L. Brylinski in [6] for Poisson manifolds; in the context of generalized complex geometry, see, e.g., [16], it can be interpreted as the symplectic counterpart of the operator $\mathrm{d}^c := -\mathrm{i}\left(\partial - \overline{\partial}\right)$ in complex geometry, see [7].
By definition, $(M,\, \omega)$ satisfies the *Hard Lefschetz Condition* if, for each $k \in \mathbb{N}$, the map

$$[\omega]^k \smile \cdot \colon H^{n-k}_{dR}(M;\mathbb{R}) \to H^{n+k}_{dR}(M;\mathbb{R})$$

is an isomorphism. O. Mathieu, [22, Corollary 2], and, independently, D. Yan, [27, Theorem 0.1], proved that, given a compact symplectic manifold $(M,\, \omega)$, any de Rham cohomology class has a (possibly non-unique) *ω-symplectically harmonic representative* (that is, a d-closed δ_ω-closed representative) if and only if the Hard Lefschetz Condition holds.

We can now prove the following result.

Theorem 2.3. *Let $(M,\, J,\, \omega,\, g)$ be a compact almost-Kähler manifold. Suppose that there exists a basis of $H^2_{dR}(X;\mathbb{R})$ represented by g-harmonic 2-forms which are of pure type with respect to J. Then the Lefschetz type property on 2-forms is satisfied.*

Proof. Recall that, on a $2n$-dimensional almost-Kähler manifold $(M,\, J,\, \omega,\, g)$, the Hodge $*_g$ operator and the symplectic $\star_\omega$ operator are related by $\star_\omega \;=\; *_g\, J$, [6, Theorem 2.4.1, Remark 2.4.4]. Therefore, for forms of pure type with respect to J, the properties of being g-harmonic and of being ω-symplectically harmonic are equivalent. The theorem follows noting that, [27, Lemma 1.2], $[\mathcal{L}_\omega,\, \mathrm{d}] = 0$ and $[\mathcal{L}_\omega,\, \delta_\omega] = \mathrm{d}$, hence $\mathcal{L}_\omega$ sends ω-symplectically harmonic 2-forms (of pure type with respect to J) to ω-symplectically harmonic $(2n-2)$-forms (of pure type with respect to J). □

Remark 2.4. We note that if $(M,\, J,\, \omega,\, g)$ is a compact $2n$-dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms and J is $\mathcal{C}^\infty$-full, then J is $\mathcal{C}^\infty$-pure-and-full and pure-and-full.

Indeed, we have already remarked that J is $\mathcal{C}^\infty$-pure, see Proposition 2.1. Moreover, since J is $\mathcal{C}^\infty$-full, J is also pure by [20, Proposition 2.5]. We recall now the argument in [15] to prove that J is also full.

Firstly, note that if the Lefschetz type property on 2-forms holds, then $\left[\omega^{n-2}\right] \smile \cdot \colon H^2_{dR}\left(M;\mathbb{R}\right) \to H^{2n-2}_{dR}\left(M;\mathbb{R}\right)$ is an isomorphism. Therefore, we get that

$$H^{2n-2}_{dR}(M;\mathbb{R}) \;=\; H^{(n,n-2),(n-2,n)}_J(M)_{\mathbb{R}} + H^{(n-1,n-1)}_J(M)_{\mathbb{R}} \;;$$

indeed, (following the argument in [15, Theorem 4.1],) since $\left[\omega^{n-2}\right] \smile \cdot \colon H^2_{dR}(M;\mathbb{R}) \to H^{2n-2}_{dR}(M;\mathbb{R})$ is in particular surjective, we have

$$\begin{aligned} H^{2n-2}_{dR}(M;\mathbb{R}) \;&=\; \left[\omega^{n-2}\right] \smile H^2_{dR}(M;\mathbb{R}) \;=\; \left[\omega^{n-2}\right] \smile \left(H^{(2,0),(0,2)}_J(M)_{\mathbb{R}} \oplus H^{(1,1)}_J(M)_{\mathbb{R}}\right) \\ &\subseteq\; H^{(n,n-2),(n-2,n)}_J(M)_{\mathbb{R}} + H^{(n-1,n-1)}_J(M)_{\mathbb{R}} \;, \end{aligned}$$

yielding the above decomposition of $H^{2n-2}_{dR}(M;\mathbb{R})$. Then, it follows that J is also full, see, for example, [1, Theorem 2.1].

2.2. **A family of almost-Kähler manifolds satisfying the Lefschetz type property on 2-forms.** Let $\mathfrak{n}$ be the 6-dimensional nilpotent Lie algebra whose structure equations, with respect to a basis $\left\{e^j\right\}_{j\in\{1,\ldots,6\}}$ of $\mathfrak{n}^*$, are given by

$$\mathrm{d}\, e^1 \;=\; \mathrm{d}\, e^2 \;=\; \mathrm{d}\, e^3 \;=\; 0 \;, \qquad \mathrm{d}\, e^4 \;=\; e^{23} \;, \qquad \mathrm{d}\, e^5 \;=\; e^{13} \;, \qquad \mathrm{d}\, e^6 \;=\; e^{12}$$

(where we write e^{jk} instead of $e^j \wedge e^k$). Using a result by Mal'tsev, [21, Theorem 7], the connected simply-connected Lie group G associated with $\mathfrak{n}$ admits a discrete co-compact subgroup Γ: let $N := \Gamma\backslash G$ be the (compact) nilmanifold obtained as a quotient of G by Γ. Note that N is not formal by a theorem of K. Hasegawa's, [17, Theorem 1, Corollary].

Fix $\alpha > 1$ and take

$$\omega_\alpha \;:=\; e^{14} + \alpha \cdot e^{25} + (\alpha - 1) \cdot e^{36} \;;$$

since $\mathrm{d}\, \omega_\alpha = 0$ and $\omega_\alpha^3 \neq 0$, we get that ω_α is a left-invariant symplectic form on N. Set

$$\begin{aligned} J_\alpha\, e_1 \;&:=\; e_4 \;, \qquad J_\alpha\, e_2 \;:=\; \alpha\, e_5 \;, \qquad J_\alpha\, e_3 \;:=\; (\alpha - 1)\, e_6 \;, \\ J_\alpha\, e_4 \;&:=\; -e_1 \;, \quad\; J_\alpha\, e_5 \;:=\; -\tfrac{1}{\alpha}\, e_2 \;, \quad J_\alpha\, e_6 \;:=\; -\tfrac{1}{\alpha - 1}\, e_3 \;, \end{aligned}$$

where $\{e_1,\, \ldots,\, e_6\}$ denotes the global dual frame of $\left\{e^1, ,\ldots,\, e^6\right\}$ on N. It is immediate to check that

- setting $g_\alpha(\cdot,\cdot) := \omega_\alpha(\cdot, J_\alpha \cdot)$, the triple $(J_\alpha,\, \omega_\alpha,\, g_\alpha)$ gives rise to a family of left-invariant almost-Kähler structures on N;
- denoting by
$$E^1_\alpha := e^1 , \qquad E^2_\alpha := \alpha\, e^2 , \qquad E^3_\alpha := (\alpha-1)\, e^3 ,$$
$$E^4_\alpha := e^4 , \qquad E^5_\alpha := e^5 , \qquad E^6_\alpha := e^6 ,$$
then $\left\{E^1_\alpha, \ldots, E^6_\alpha\right\}$ is a g_α-orthonormal co-frame on N; with respect to this new co-frame, we easily obtain the following structure equations:

$$\mathrm{d}\, E^1_\alpha = \mathrm{d}\, E^2_\alpha = \mathrm{d}\, E^3_\alpha = 0, \quad \mathrm{d}\, E^4_\alpha = \frac{1}{\alpha(\alpha-1)}\, E^{23}_\alpha, \quad \mathrm{d}\, E^5_\alpha = \frac{1}{\alpha-1}\, E^{13}_\alpha, \quad \mathrm{d}\, E^6_\alpha = \frac{1}{\alpha}\, E^{12}_\alpha.$$

Then,
$$\varphi^1_\alpha := E^1_\alpha + \mathrm{i}\, E^4_\alpha , \qquad \varphi^2_\alpha := E^2_\alpha + \mathrm{i}\, E^5_\alpha , \qquad \varphi^3_\alpha := E^3_\alpha + \mathrm{i}\, E^6_\alpha ,$$
are $(1,0)$-forms with respect to the almost-complex structure J_α, and
$$\omega_\alpha \;=\; E^{14}_\alpha + E^{25}_\alpha + E^{36}_\alpha .$$

By a result of K. Nomizu's, [25, Theorem 1], see Theorem 5.3, the de Rham cohomology of N is straightforwardly computed:
$$H^2_{dR}(N;\mathbb{R}) \;\simeq\; \mathbb{R}\left\langle E^{15}_\alpha,\, E^{16}_\alpha,\, E^{24}_\alpha,\, E^{26}_\alpha,\, E^{34}_\alpha,\, E^{35}_\alpha,\, E^{14}_\alpha + \frac{1}{\alpha}\, E^{25}_\alpha,\, \frac{1}{\alpha}\, E^{25}_\alpha + \frac{1}{\alpha-1}\, E^{36}_\alpha \right\rangle$$
(where we have listed the g_α-harmonic representatives instead of their classes). Note that the listed g_α-harmonic representatives of $H^2_{dR}(N;\mathbb{R})$ are of pure type with respect to J_α: hence, the almost-complex structure J_α is $\mathcal{C}^\infty$-pure-and-full by [15, Theorem 3.7]; in particular, note that
$$\begin{aligned} H^2_{dR}(N;\mathbb{R}) \;\simeq\;& \mathbb{R}\left\langle \mathrm{i}\, \alpha\, \varphi^{1\bar1}_\alpha + \mathrm{i}\, \varphi^{2\bar2}_\alpha,\, \mathrm{i}\,(\alpha-1)\, \varphi^{2\bar2}_\alpha + \mathrm{i}\, \alpha\, \varphi^{3\bar3}_\alpha,\, \Im\mathfrak{m}\, \varphi^{1\bar2}_\alpha,\, \Im\mathfrak{m}\, \varphi^{1\bar3}_\alpha,\, \Im\mathfrak{m}\, \varphi^{3\bar2}_\alpha \right\rangle \\ & \oplus \left\langle \Im\mathfrak{m}\, \varphi^{12}_\alpha,\, \Im\mathfrak{m}\, \varphi^{13}_\alpha,\, \Im\mathfrak{m}\, \varphi^{23}_\alpha \right\rangle , \end{aligned}$$
hence $h^+_{J_\alpha}(N) = 5$ and $h^-_{J_\alpha}(N) = 3$.

Moreover, one explicitly notes that
$$\begin{array}{lllll} \mathcal{L}_{\omega_\alpha} E^{15}_\alpha = E^{1536}_\alpha = *_{g_\alpha} E^{24}_\alpha , & \qquad & \mathcal{L}_{\omega_\alpha} E^{16}_\alpha = E^{1625}_\alpha = *_{g_\alpha} E^{34}_\alpha , \\ \mathcal{L}_{\omega_\alpha} E^{24}_\alpha = E^{2436}_\alpha = *_{g_\alpha} E^{15}_\alpha , & \qquad & \mathcal{L}_{\omega_\alpha} E^{26}_\alpha = E^{2614}_\alpha = *_{g_\alpha} E^{35}_\alpha , \\ \mathcal{L}_{\omega_\alpha} E^{34}_\alpha = E^{3425}_\alpha = *_{g_\alpha} E^{16}_\alpha , & \qquad & \mathcal{L}_{\omega_\alpha} E^{35}_\alpha = E^{3514}_\alpha = *_{g_\alpha} E^{26}_\alpha , \end{array}$$
while
$$\mathcal{L}_{\omega_\alpha} \left(E^{14}_\alpha + \frac{1}{\alpha}\, E^{25}_\alpha\right) \;=\; -\frac{\alpha+1}{\alpha}\, E^{1245}_\alpha - \frac{1}{\alpha}\, E^{2356}_\alpha - E^{1346}_\alpha$$
where
$$\mathrm{d} *_{g_\alpha} \mathcal{L}_{\omega_\alpha} \left(E^{14}_\alpha + \frac{1}{\alpha}\, E^{25}_\alpha\right) \;=\; \mathrm{d}\left(-\frac{\alpha+1}{\alpha}\, E^{36}_\alpha - E^{25}_\alpha - \frac{1}{\alpha}\, E^{14}_\alpha\right) = 0 ,$$
and, by a similar computation, $\mathrm{d} *_{g_\alpha} \mathcal{L}_{\omega_\alpha} \left(e^{25} + e^{36}\right) = 0$. This proves explicitly that ω_α satisfies the Lefschetz type property on 2-forms.

The nilmanifold N is not formal by a theorem of K. Hasegawa's, [17, Theorem 1, Corollary]. The non-formality of M can be also proved by giving a non-zero triple Massey product on N, see [9]: since
$$\left[E^1_\alpha\right] \smile \left[E^3_\alpha\right] \;=\; (\alpha-1)\,\left[\mathrm{d}\, E^5_\alpha\right] \;=\; 0 , \quad \left[E^3_\alpha\right] \smile \left[E^2_\alpha\right] \;=\; -\alpha\,(\alpha-1)\,\left[\mathrm{d}\, E^4_\alpha\right] \;=\; 0 ,$$

we get that the triple Massey product

$$\left\langle [E^1_\alpha],\, [E^3_\alpha],\, [E^2_\alpha] \right\rangle \;=\; -\,(\alpha-1)\;\left[E^{25}_\alpha + \alpha\, E^{14}_\alpha\right]$$

does not vanish, and hence N is not formal.

In summary, we have proven the following result.

Proposition 2.5. *There is a non-formal 6-dimensional nilmanifold N endowed with a 1-parameter family $\{(J_\alpha,\, \omega_\alpha,\, g_\alpha)\}_{\alpha>1}$ of left-invariant almost-Kähler structures being $\mathcal{C}^\infty$-pure-and-full and pure-and-full and satisfying the Lefschetz type property on 2-forms.*

Remark 2.6. It has to be noted that $\omega_\alpha \wedge \cdot\colon\ \wedge^2 N^6 \to \wedge^4 N^6$ induces an isomorphism in cohomology $[\omega_\alpha] \smile \cdot\colon H^2_{dR}(N,\mathbb{R}) \to H^4_{dR}(N,\mathbb{R})$, while, accordingly to [5, Theorem A], $[\omega_\alpha]^2 \smile \cdot\colon H^1_{dR}(N,\mathbb{R}) \to H^5_{dR}(N,\mathbb{R})$ is not an isomorphism.

3. Almost-Kähler $\mathcal{C}^\infty$-pure-and-full structures

3.1. **The Nakamura manifold of completely solvable type.** Take $A \in \mathrm{SL}(2;\mathbb{Z})$ with two different real eigenvalues e^λ and $\mathrm{e}^{-\lambda}$ with $\lambda > 0$, and fix $P \in \mathrm{GL}(2;\mathbb{R})$ such that $PAP^{-1} = \mathrm{diag}\left(\mathrm{e}^\lambda,\, \mathrm{e}^{-\lambda}\right)$. For example, take

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad P := \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

and consequently $\lambda = \log \frac{3+\sqrt{5}}{2}$. Let $M^6 =: M^6(\lambda)$ be the compact complex manifold

$$M^6 := \mathbb{S}^1_{x^2} \times \frac{\mathbb{R}_{x^1} \times \mathbb{T}^2_{\mathbb{C},\,(x^3,\,x^4,\,x^5,\,x^6)}}{\langle T_1 \rangle}$$

where $\mathbb{T}^2_{\mathbb{C}}$ is the 2-dimensional complex torus $\mathbb{T}^2_{\mathbb{C}} := \frac{\mathbb{C}^2}{P\mathbb{Z}[\mathrm{i}]^2}$ and T_1 acts on $\mathbb{R} \times \mathbb{T}^2_{\mathbb{C}}$ as $T_1\left(x^1,\, x^3,\, x^4,\, x^5,\, x^6\right) := \left(x^1+\lambda,\, \mathrm{e}^{-\lambda}\, x^3,\, \mathrm{e}^{\lambda}\, x^4,\, \mathrm{e}^{-\lambda}\, x^5,\, \mathrm{e}^{\lambda}\, x^6\right)$. The manifold M^6 can be seen as a compact quotient of a completely-solvable Lie group by a discrete co-compact subgroup, [14, Example 3.1]; (denote the Lie algebra naturally associated to the completely-solvable Lie group of M^6 by $\mathfrak{g}$). Using coordinates x^2 on $\mathbb{S}^1$, x^1 on $\mathbb{R}$ and $\left(x^3, x^4, x^5, x^6\right)$ on $\mathbb{T}^2_{\mathbb{C}}$, we set

$$e^1 := \mathrm{d}\,x^1,\ \ e^2 := \mathrm{d}\,x^2,\ \ e^3 := \mathrm{e}^{x^1}\,\mathrm{d}\,x^3,\ \ e^4 := \mathrm{e}^{-x^1}\,\mathrm{d}\,x^4,\ \ e^5 := \mathrm{e}^{x^1}\,\mathrm{d}\,x^5,\ \ e^6 := \mathrm{e}^{-x^1}\,\mathrm{d}\,x^6$$

as a basis for $\mathfrak{g}^*$; therefore, with respect to $\{e^i\}_{i\in\{1,\ldots,6\}}$, the structure equations are the following:

$$\mathrm{d}\,e^1 \;=\; \mathrm{d}\,e^2 \;=\; 0\,, \quad \mathrm{d}\,e^3 \;=\; e^{13}\,, \quad \mathrm{d}\,e^4 \;=\; -e^{14}\,, \quad \mathrm{d}\,e^5 \;=\; e^{15}\,, \quad \mathrm{d}\,e^6 \;=\; -e^{16}\,.$$

3.2. **The de Rham cohomology of the Nakamura manifold.** Let J be the almost-complex structure on M^6 defined by the complex $(1,0)$-forms given by

$$\varphi^1 \;:=\; \frac{1}{2}\left(e^1 + \mathrm{i}\,e^2\right)\,, \qquad \varphi^2 \;:=\; e^3 + \mathrm{i}\,e^5\,, \qquad \varphi^3 \;:=\; e^4 + \mathrm{i}\,e^6\,.$$

It is straightforward to check that J is integrable.

Being M^6 a compact quotient of a completely-solvable Lie group, one computes the de Rham cohomology of M^6 easily by A. Hattori's theorem [19, Corollary 4.2], see Theorem 5.3:

$$H^1_{dR}\left(M^6;\mathbb{C}\right) \simeq \mathbb{C}\left\langle\varphi^1,\,\bar\varphi^1\right\rangle\,, \quad H^2_{dR}\left(M^6;\mathbb{C}\right) \simeq \mathbb{C}\left\langle\varphi^{1\bar1},\,\varphi^{2\bar3},\,\varphi^{3\bar2},\,\varphi^{23},\,\varphi^{\bar2\bar3}\right\rangle\,,$$
$$H^3_{dR}\left(M^6;\mathbb{C}\right) \simeq \mathbb{C}\left\langle\varphi^{12\bar3},\,\varphi^{13\bar2},\,\varphi^{123},\,\varphi^{1\bar2\bar3},\,\varphi^{2\bar1\bar3},\,\varphi^{3\bar1\bar2},\,\varphi^{23\bar1},\,\varphi^{\bar1\bar2\bar3}\right\rangle$$

(for the sake of clearness, we write, for example, $\varphi^{A\bar B}$ in place of $\varphi^A\wedge\bar\varphi^B$ and we list the harmonic representatives with respect to the metric $g:=\sum_{j=1}^3\varphi^j\odot\bar\varphi^j$ instead of their classes). Therefore, M^6 is *geometrically formal*, i.e., the product of g-harmonic forms is still g-harmonic, and therefore it is *formal*, namely the de Rham complex of M is formal as a differential graded algebra, see, e.g., [9]. Furthermore, it can be easily checked that

$$\omega := e^{12}+e^{34}+e^{56}$$

gives rise to a symplectic structure on M^6 satisfying the Hard Lefschetz Condition. We obtain the following result.

Proposition 3.1 ([14, Proposition 3.2]). *The manifold M^6 is formal and it admits a symplectic form ω satisfying the Hard Lefschetz Condition.*

Note also that $\tilde\omega := \frac{\mathrm{i}}{2}\left(\varphi^{1\bar1}+\varphi^{2\bar2}+\varphi^{3\bar3}\right)$ is not d-closed but $\mathrm{d}\,\tilde\omega^2=0$, from which it follows that the manifold M^6 admits a balanced metric.

Moreover, since M^6 is a compact quotient of a completely-solvable Lie group, by the K. Hasegawa's theorem [18, Main Theorem], we have the following result, see also [14, Theorem 3.3]. (We recall that a compact complex manifold is said to belong to *class $\mathcal{C}$ of Fujiki* if it admits a proper modification from a Kähler manifold.)

Theorem 3.2 ([18, Main Theorem]). *The manifold M^6 admits no Kähler structure and it is not in class $\mathcal{C}$ of Fujiki.*

3.3. **An almost-Kähler structure on the Nakamura manifold.** By K. Hasegawa's theorem [18, Main Theorem], any integrable complex structure on M^6 (for example, the J defined in §3.2) does not admit any symplectic structure compatible with it. Therefore, we consider the almost-complex structure J' defined by

$$J'e^1 := -e^2\,, \qquad J'e^3 := -e^4\,, \qquad J'e^5 := -e^6\,;$$

considering

$$\psi^1 := \frac{1}{2}\left(e^1+\mathrm{i}\,e^2\right)\,, \quad \psi^2 := e^3+\mathrm{i}\,e^4\,, \quad \psi^3 := e^5+\mathrm{i}\,e^6$$

as a co-frame for the space of $(1,0)$-forms on $\left(M^6,\,J'\right)$, one can compute

$$\mathrm{d}\,\psi^1=0\,,\quad \mathrm{d}\,\psi^2=\psi^{1\bar2}+\psi^{\bar1\bar2}\,,\quad \mathrm{d}\,\psi^3=\psi^{1\bar3}+\psi^{\bar1\bar3}\,,$$

from which it is clear that J' is not integrable. Note that the J'-compatible 2-form

$$\omega' := e^{12}+e^{34}+e^{56}$$

is d-closed. Hence, $\left(M^6,\,J',\,\omega'\right)$ is an almost-Kähler manifold.
Moreover, recall that

$$H^2_{dR}\left(M^6;\mathbb{R}\right) \simeq \underbrace{\mathbb{R}\left\langle\mathrm{i}\,\psi^{1\bar1},\,\mathrm{i}\,\psi^{2\bar2},\,\mathrm{i}\,\psi^{3\bar3},\,\mathrm{i}\left(\psi^{2\bar3}+\psi^{3\bar2}\right)\right\rangle}_{\subseteq H^+_{J'}(M^6)_{\mathbb{R}}} \oplus \underbrace{\mathbb{R}\left\langle\mathrm{i}\left(\psi^{23}-\psi^{\bar2\bar3}\right)\right\rangle}_{\subseteq H^-_{J'}(M^6)_{\mathbb{R}}}\,,$$

where we have listed the harmonic representatives with respect to the metric $g' := \sum_{j=1}^{6} e^j \odot e^j$ instead of their classes; note that the listed g'-harmonic representatives are of pure type with respect to J'. Therefore, J' is obviously $\mathcal{C}^\infty$-full; it is also $\mathcal{C}^\infty$-pure by [15, Proposition 3.2] or [11, Proposition 2.8], see Proposition 2.1. Moreover, since any cohomology class in $H^+_{J'}\left(M^6\right)_{\mathbb{R}}$ (respectively, in $H^-_{J'}\left(M^6\right)_{\mathbb{R}}$) has a g'-harmonic representative in $\mathcal{Z}^{(1,1)}_{J'}$ (respectively, in $\mathcal{Z}^{(2,0),(0,2)}_{J'}$), by [15, Theorem 3.7] we have that J' is also pure-and-full. One can explicitly check that the Lefschetz type operator $\mathcal{L}_{\omega'}\colon \wedge^2 M^6 \to \wedge^4 M^6$ introduced in §2 takes g'-harmonic 2-forms to g'-harmonic 4-forms, since

$$\begin{array}{ll} \mathcal{L}_{\omega'}\, e^{12} = e^{1234} + e^{1256} = *_{g'}\left(e^{34} + e^{56}\right) , & \mathcal{L}_{\omega'}\, e^{36} = e^{1236} = *_{g'}\, e^{45} , \\ \mathcal{L}_{\omega'}\, e^{34} = e^{1234} + e^{3456} = *_{g'}\left(e^{12} + e^{56}\right) , & \mathcal{L}_{\omega'}\, e^{45} = e^{1245} = *_{g'}\, e^{36} , \\ \mathcal{L}_{\omega'}\, e^{56} = e^{1256} + e^{3456} = *_{g'}\left(e^{12} + e^{34}\right) . & \end{array}$$

Resuming, we have shown the following result.

Proposition 3.3. *Let M^6 be the Nakamura manifold. Then there exist a complex structure J and an almost-Kähler structure $(J',\, \omega',\, g')$, both of which are $\mathcal{C}^\infty$-pure-and-full and pure-and-full.*
Furthermore, the Lefschetz type operator of the almost-Kähler structure $(J',\, \omega',\, g')$ takes g'-harmonic 2-forms to g'-harmonic 4-forms.

Inspired by the argument of the proof of [11, Theorem 2.3], see Theorem 1.3, one can ask the following question, compare also [13, §2]; we provide in Proposition 4.1 an example of a non-$\mathcal{C}^\infty$-full almost-Kähler structure for which the Lefschetz type property on 2-forms does not hold.

Question 3.4. Let $(M,\, J,\, \omega,\, g)$ be a compact $2n$-dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms. Is J $\mathcal{C}^\infty$-full?

4. An almost-Kähler non-$\mathcal{C}^\infty$-full structure

Let $X := \mathbb{Z}[\mathrm{i}]^3 \big\backslash \left(\mathbb{C}^3,\, *\right)$ be the *Iwasawa manifold*, where the group structure on $\mathbb{C}^3$ is defined by

$$(z_1,\, z_2,\, z_3) * (w_1,\, w_2,\, w_3) := (z_1 + w_1,\, z_2 + w_2,\, z_3 + z_1 w_2 + w_3) \ .$$

Considering the standard complex structure induced by the one on $\mathbb{C}^3$ and setting $\left\{\varphi^1,\, \varphi^2,\, \varphi^3\right\}$ as a global co-frame for the $(1,0)$-forms on X, by A. Hattori's theorem [19, Corollary 4.2], see Theorem 5.3, one gets that

$$\begin{aligned} H^2_{dR}(X;\mathbb{C}) \ \simeq\ \ & \mathbb{R}\left\langle \varphi^{13} + \varphi^{\bar1\bar3},\ \mathrm{i}\left(\varphi^{13} - \varphi^{\bar1\bar3}\right),\ \varphi^{23} + \varphi^{\bar2\bar3},\ \mathrm{i}\left(\varphi^{23} - \varphi^{\bar2\bar3}\right),\right. \\ & \left. \varphi^{1\bar2} - \varphi^{2\bar1},\ \mathrm{i}\left(\varphi^{1\bar2} + \varphi^{2\bar1}\right),\ \mathrm{i}\,\varphi^{1\bar1},\ \mathrm{i}\,\varphi^{2\bar2} \right\rangle \otimes_{\mathbb{R}} \mathbb{C} , \end{aligned}$$

where we have listed the harmonic representatives with respect to the metric $g := \sum_{h=1}^{3} \varphi^h \odot \bar\varphi^h$ instead of their classes. Set

$$\varphi^1 =: e^1 + \mathrm{i}\, e^2 , \qquad \varphi^2 =: e^3 + \mathrm{i}\, e^4 , \qquad \varphi^3 =: e^5 + \mathrm{i}\, e^6 ;$$

then,

$$\mathrm{d}\, e^5 \ =\ -e^{13} + e^{24} , \qquad \mathrm{d}\, e^6 \ =\ -e^{14} - e^{23} ,$$

the other differentials being zero. Therefore,

$$H^2_{dR}(X;\mathbb{R}) \ \simeq\ \mathbb{R}\left\langle e^{15} - e^{26},\, e^{16} + e^{25},\, e^{35} - e^{46},\, e^{36} + e^{45},\, e^{13} + e^{24},\, e^{23} - e^{14},\, e^{12},\, e^{34} \right\rangle \ .$$

Set

$$v_1 := e^{15} - e^{26} , \qquad v_2 := e^{16} + e^{25} , \qquad v_3 := e^{35} - e^{46} , \qquad v_4 := e^{36} + e^{45} ,$$
$$v_5 := e^{13} + e^{24} , \qquad v_6 := e^{23} - e^{14} , \qquad v_7 := e^{12} , \qquad v_8 := e^{34} .$$

Consider the almost-Kähler structure (J, ω, g) on X defined by

$$Je^1 := -e^6 , \qquad Je^2 := -e^5 , \qquad Je^3 := -e^4 , \qquad \omega := e^{16} + e^{25} + e^{34} .$$

We easily get that

$$\mathbb{R}\langle v_2, v_3 + v_5, v_4 - v_6, v_8 \rangle \subseteq H^+_J(X) , \qquad \mathbb{R}\langle v_1, v_3 - v_5, v_4 + v_6 \rangle \subseteq H^-_J(X) .$$

We claim that the previous inclusions are actually equalities, and in particular that *J is a non-$\mathcal{C}^\infty$-full almost-Kähler structure* on X.
Indeed, we firstly note that, by [15, Proposition 3.2] or [11, Proposition 2.8], see Proposition 2.1, J is $\mathcal{C}^\infty$-pure, since it admits a symplectic structure compatible with it. Moreover, we recall that a $\mathcal{C}^\infty$-full almost-complex structure is also pure by [20, Proposition 2.30] and therefore it satisfies also that

$$H^{(3,1),(1,3)}_J(X)_\mathbb{R} \cap H^{(2,2)}_J(X)_\mathbb{R} = \{0\} , \tag{1}$$

see [1, Theorem 2.4]. Therefore, our claim reduces to prove that J does not satisfy (1). Note that

$$\begin{aligned}\left[e^{3456}\right] &= \left[e^{3456} - \mathrm{d}\, e^{135}\right] = \left[e^{3456} + e^{1234}\right] \\ &= \left[e^{3456} + \mathrm{d}\, e^{135}\right] = \left[e^{3456} - e^{1234}\right]\end{aligned}$$

and that $e^{3456} + e^{1234} \in \wedge^{(3,1),(1,3)}_J(X)_\mathbb{R}$ while $e^{3456} - e^{1234} \in \wedge^{(2,2)}_J(X)_\mathbb{R}$, and so $H^{(3,1),(1,3)}_J(X)_\mathbb{R} \cap H^{(2,2)}_J(X)_\mathbb{R} \ni \left[e^{3456}\right]$, therefore (1) does not hold, and hence J is not $\mathcal{C}^\infty$-full.

Let $\mathcal{L}_\omega$ be the Lefschetz type operator of the almost-Kähler structure (J, ω, g). Then, we have $\mathcal{L}_\omega\left(e^{12}\right) = e^{1234} = \mathrm{d}\left(e^{245}\right)$, i.e., $\mathcal{L}_\omega$ does not take g-harmonic 2-forms in g-harmonic 4-forms.

Hence, we have proved the following result.

Proposition 4.1. *Let $X := \mathbb{Z}[\mathrm{i}]^3 \big\backslash \left(\mathbb{C}^3, *\right)$ be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure (J, ω, g) on X which is $\mathcal{C}^\infty$-pure and non-$\mathcal{C}^\infty$-full.*
Furthermore, the Lefschetz type operator of the almost-Kähler structure (J, ω, g) does not take g-harmonic 2-forms to g-harmonic 4-forms.

5. Almost-complex manifolds with large anti-invariant cohomology

Given an almost-complex structure J on a compact manifold M, it is natural to ask how large the cohomology subgroup $H^{(2,0),(0,2)}_J(M)_\mathbb{R}$ can be. In this direction, T. Drăghici, T.-J. Li, and the third author raised the following question in [12].

Question 5.1 ([12, Conjecture 2.5])**.** Are there compact 4-dimensional manifold M endowed with non-integrable almost-complex structures J such that $\dim_\mathbb{R} H^-_J(M) \geq 3$?

We present here a 1-parameter family $\{J_t\}_t$ of (non-integrable) almost-complex structures on the 6-dimensional torus $\mathbb{T}^6$ having $h^-_{J_t} := \dim_{\mathbb{R}} H^-_{J_t}\left(\mathbb{T}^6\right)_{\mathbb{R}}$ greater than 3, see also [1, §4]. For t small enough, set $\alpha_t :=: \alpha_t\left(x^3\right) \in \mathcal{C}^\infty\left(\mathbb{T}^6\right)$ such that $\alpha_0(x^3) \equiv 1$ and set

$$\varphi^1_t \;:=\; \mathrm{d}\,x^1 + \mathrm{i}\,\alpha_t\,\mathrm{d}\,x^4\,, \quad \varphi^2_t \;:=\; \mathrm{d}\,x^2 + \mathrm{i}\,\mathrm{d}\,x^5\,, \quad \varphi^3_t \;:=\; \mathrm{d}\,x^3 + \mathrm{i}\,\mathrm{d}\,x^6\,;$$

therefore, the structure equations are

$$\mathrm{d}\,\varphi^1_t \;=\; \mathrm{i}\,\mathrm{d}\,\alpha_t \wedge \mathrm{d}\,x^4\,, \quad \mathrm{d}\,\varphi^2_t \;=\; 0\,, \quad \mathrm{d}\,\varphi^3_t \;=\; 0\,.$$

Straightforward computations give that the J-anti-invariant d-closed 2-forms are of the type

$$\psi \;=\; \frac{C}{\alpha_t}\left(\mathrm{d}\,x^{13} - \alpha_t\,\mathrm{d}\,x^{46}\right) + D\left(\mathrm{d}\,x^{16} - \alpha_t\,\mathrm{d}\,x^{34}\right) + E\left(\mathrm{d}\,x^{23} - \mathrm{d}\,x^{56}\right) + F\left(\mathrm{d}\,x^{26} - \mathrm{d}\,x^{35}\right)\,,$$

where $C,\ D,\ E,\ F \in \mathbb{R}$ (we shorten $\mathrm{d}\,x^j \wedge \mathrm{d}\,x^k$ by $\mathrm{d}\,x^{jk}$). Moreover, the forms $\mathrm{d}\,x^{23} - \mathrm{d}\,x^{56}$ and $\mathrm{d}\,x^{26} - \mathrm{d}\,x^{35}$ are clearly harmonic with respect to the standard flat metric $\sum_{j=1}^{6} \mathrm{d}\,x^j \otimes \mathrm{d}\,x^j$, while the classes of $\mathrm{d}\,x^{16} - \alpha_t\,\mathrm{d}\,x^{34}$ and $\mathrm{d}\,x^{13} - \alpha_t\,\mathrm{d}\,x^{46}$ are non-zero, their harmonic parts being non-zero. Hence, we get that $h^-_{J_0} = 6$ and

$$h^-_{J_t} \;=\; 4 \qquad \text{for small} \qquad t \neq 0\,.$$

In the general case, we ask the following natural question.

Question 5.2. Are there examples of non-integrable almost-complex structures J on a compact $2n$-dimensional manifold with $\dim_{\mathbb{R}} H^-_J(M) > n\,(n-1)$?

Consider now a solvmanifold $M = \Gamma\backslash G$, namely, a compact quotient of a connected simply-connected solvable Lie group G by a co-compact discrete subgroup Γ. Denote the Lie algebra naturally associated to G by $\mathfrak{g}$, and consider $(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d})$ the subcomplex of the de Rham complex $(\wedge^\bullet M,\, \mathrm{d})$ given by the left-invariant differential forms. The following result by K. Nomizu [25] and A. Hattori [19] holds.

Theorem 5.3 ([25, Theorem 1], [19, Theorem 4.2])**.** *Let M be a nilmanifold or, more in general, a completely-solvable solvmanifold. Then $H^\bullet\left(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}\right) \simeq H^\bullet_{dR}(M;\mathbb{R})$.*

Let J be a left-invariant almost-complex structure on M, namely, an almost-complex structure on M induced by an almost-complex structure on G that is invariant under the action of G on itself given by left-translations. Given $p, q \in \mathbb{N}$, denote by

$$H^{(p,q),(q,p)}_J(\mathfrak{g})_{\mathbb{R}} \;:=\; \left\{\mathfrak{a} = [\alpha] \in H^\bullet\left(\wedge^\bullet\mathfrak{g}^*,\, \mathrm{d}\right) \;:\; \alpha \in \wedge^{(p,q),(q,p)}_J \mathfrak{g}^*\right\} \;\subseteq\; H^\bullet_{dR}(M;\mathbb{R})$$

the subgroup (see, e.g., [8, Lemma 9]) of $H^\bullet_{dR}(M;\mathbb{R})$ that consists of classes admitting a left-invariant representative of type $(p,q)+(q,p)$, where $\wedge^{(p,q),(q,p)}_J\mathfrak{g}^* := \left(\wedge^{p,q}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^* \oplus \wedge^{q,p}\left(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\right)^*\right) \cap \wedge^\bullet\mathfrak{g}^*$.

Using Belgun's symmetrization trick, [4, Theorem 7], one can prove the following Nomizu-type result, which relates the subgroups $H^{(p,q),(q,p)}_J(M)_{\mathbb{R}}$ with their left-invariant part $H^{(p,q),(q,p)}_J(\mathfrak{g})_{\mathbb{R}}$.

Theorem 5.4. *Let $M = \Gamma \backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. For any $p, q \in \mathbb{N}$, the map*

$$j \colon H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$$

induced by left-translations is injective, and, if $H^\bullet_{dR}(\wedge^\bullet \mathfrak{g}^*, \mathrm{d}) \simeq H^\bullet_{dR}(M;\mathbb{R})$ (for instance, if M is a completely-solvable solvmanifold), then $j \colon H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ is in fact an isomorphism.

Proof. Since J is left-invariant, left-translations induce the map $j \colon H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$.

Since, by J. Milnor's Lemma [24, Lemma 6.2], G is unimodular, one can take in particular a bi-invariant volume form η on M such that $\int_M \eta = 1$. Consider the F. A. Belgun symmetrization map in [4, Theorem 7], namely,

$$\mu \colon \wedge^\bullet M \to \wedge^\bullet \mathfrak{g}^* , \qquad \mu(\alpha) := \int_M \alpha\lfloor_m \eta(m) .$$

Since μ commutes with d by [4, Theorem 7], it induces the map $\mu \colon H^\bullet_{dR}(M;\mathbb{R}) \to H^\bullet(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$, and, since μ commutes with J, it preserves the bi-graduation; therefore it induces the map $\mu \colon H_J^{(p,q),(q,p)}(M) \to H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$. Moreover, since μ is the identity on the space of left-invariant forms by [4, Theorem 7], we get the commutative diagram

$$\begin{tikzcd} H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \arrow[r, "\mu"] & H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \end{tikzcd}$$

hence $j \colon H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}$ is injective, and $\mu \colon H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$ is surjective.

Furthermore, when $H^\bullet(\wedge^\bullet \mathfrak{g}^*, \mathrm{d}) \simeq H^\bullet_{dR}(M;\mathbb{R})$ (for instance, when M is a completely-solvable solvmanifold, by A. Hattori's theorem [19, Theorem 4.2], see Theorem 5.3), since $\mu\lfloor_{\wedge^\bullet \mathfrak{g}^*} = \mathrm{id}\lfloor_{\wedge^\bullet \mathfrak{g}^*}$ by [4, Theorem 7], we get that $\mu \colon H^\bullet_{dR}(M;\mathbb{R}) \to H^\bullet(\wedge^\bullet \mathfrak{g}^*, \mathrm{d})$ is the identity map, and hence $\mu \colon H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} \to H_J^{(p,q),(q,p)}(\mathfrak{g})_{\mathbb{R}}$ is also injective, and hence an isomorphism. □

In particular, if $M = \Gamma \backslash G$ is a $2n$-dimensional completely-solvable solvmanifold endowed with a left-invariant almost-complex structure J, then

$$\dim_{\mathbb{R}} H_J^-(M) \leq n\,(n-1) \qquad \text{and} \qquad \dim_{\mathbb{R}} H_J^+(M) \leq n^2 ;$$

this provides a partial negative answer to Question 5.2.

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(Daniele Angella) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127, Pisa, Italy
E-mail address: `angella@mail.dm.unipi.it`

(Adriano Tomassini) Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124, Parma, Italy
E-mail address: `adriano.tomassini@unipr.it`

(Weiyi Zhang) University of Michigan, Department of Mathematics, 1825 East Hall, Ann Arbor, MI 48109
E-mail address: `wyzhang@umich.edu`