ON COHOMOLOGICAL DECOMPOSABILITY OF ALMOST–KÄHLER STRUCTURES

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Abstract. We study the $J$-invariant and $J$-anti-invariant cohomological subgroups of the de Rham cohomology of a compact manifold $M$ endowed with an almost-Kähler structure $(J, \omega, g)$. In particular, almost-Kähler manifolds satisfying a Lefschetz type property, and solvmanifolds endowed with left-invariant almost-complex structures are investigated.


Introduction

Cohomological properties of compact complex, and, more in general, almost-complex, manifolds have been recently studied by many authors, see, e.g., [3], respectively [11, 12], and the references therein. The study of the cohomology of almost-complex manifolds is motivated, in particular, by a question of Donaldson’s, [10, Question 2], relating the tamed and compatible symplectic cones of a compact 4-dimensional almost-complex manifold, see, e.g., [20], and by the analogous question arising for compact higher dimensional complex manifolds, see [20, page 678] and [26, Question 1.7]. (We recall that a symplectic structure $\omega$ on a manifold $M$ is said to tame an almost-complex structure $J$ if $\omega_x (u_x, J_x u_x) > 0$ for any $x \in M$ and for any $u \in T_x M \setminus \{0\}$, and it is said compatible with $J$ if $g := \omega(\cdot, J \cdot \cdot)$ is a $J$-Hermitian metric; in the latter case, the triple $(J, \omega, g)$ is called an almost-Kähler structure on $M$.)

Following T.-J. Li and the third author, [20], an almost-complex structure $J$ on a $2n$-dimensional manifold $M$ is called $C^\infty$-pure-and-full if

$$H^{2}_{dR}(M; \mathbb{R}) = H^{(1,1)}_{J}(M)_{\mathbb{R}} \oplus H^{(2),0,(0,2)}_{J}(M)_{\mathbb{R}},$$

where $H^{(1,1)}_{J}(M)_{\mathbb{R}}$ and $H^{(2),0,(0,2)}_{J}(M)_{\mathbb{R}}$ denote the subgroups of $H^{2}_{dR}(M; \mathbb{R})$ whose elements can be represented by forms of type $(1, 1)$ and $(2, 0) + (0, 2)$ respectively. In the notation of T. Drăghici, T.-J. Li, and the third author, [11], $H^{(1,1)}_{J}(M)_{\mathbb{R}} = H^{J}_{+}(M)$ and $H^{(2),0,(0,2)}_{J}(M)_{\mathbb{R}} = H^{J}_{-}(M)$ are the $J$-invariant and the $J$-anti-invariant cohomology subgroups respectively.

In [11, Theorem 2.3], T. Drăghici, T.-J. Li, and the third author proved that every almost-complex structure on a compact 4-dimensional manifold is $C^{\infty}$-pure-and-full. This is no more true in dimension greater than four, see, e.g., [15, Example 3.3], see also [1] [2].

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The groups $H^{(1,1)}_J(M)_\mathbb{R}$ and $H^{(2,0),(0,2)}_J(M)_\mathbb{R}$ appear as a natural generalization of the Dolbeault cohomology groups to the non-integrable case, see, e.g., [20 Proposition 2.1]. In fact, compact Kähler manifolds are $C^\infty$-pure-and-full, and, in this case, $H^{(1,1)}_J(M)_\mathbb{R} \simeq H^1_\mathbb{R}(M) \cap H^2_{dR}(M; \mathbb{R})$ and $H^{(2,0),(0,2)}_J(M)_\mathbb{R} \simeq \left( H^2_{dR}(M) \oplus H^2_{\mathbb{C}}(M) \right) \cap H^2_{dR}(M; \mathbb{R})$.

We remark that, on a compact complex manifold, other cohomologies can be defined, namely, the Bott-Chern and Aeppli cohomologies. In [3], the problem of cohomology decomposition in terms of the Bott-Chern cohomology groups is investigated, providing in particular a characterization of compact complex manifolds satisfying the $\partial\bar{\partial}$-Lemma.

Compact Kähler manifolds being $C^\infty$-pure-and-full, in this paper we are interested in the study of the cohomological subgroups $H^{(1,1)}_J(M)_\mathbb{R}$ and $H^{(2,0),(0,2)}_J(M)_\mathbb{R}$ for almost-Kähler manifolds.

On the one hand, A. Fino and the second author, [14 Proposition 3.2], as well as T. Drăghici, T.-J. Li, and the third author, [11 Proposition 2.8], proved that the almost-complex structure of a compact almost-Kähler manifold is $C^\infty$-pure. On the other hand, we prove the following result, showing therefore a difference between the integrable and the non-integrable cases.

**Proposition 4.1.** Let $X := Z[\mathbb{H}^3 \setminus (\mathbb{C}^3, *)]$ be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure $(J, \omega, g)$ on $X$ which is $C^\infty$-pure and non-$C^\infty$-full. Furthermore, the Lefschetz type operator $\mathcal{L}_\omega := \omega \wedge \cdot : \wedge^2 M \to \wedge^4 M$ of the almost-Kähler structure $(J, \omega, g)$ does not take $g$-harmonic 2-forms to $g$-harmonic 4-forms.

In studying cohomological decomposition of the de Rham cohomology of almost-Kähler manifolds, the third author introduced a Lefschetz type property for 2-forms, see Definition 2.2. Such a property is stronger than the Hard Lefschetz Condition on 2-classes, namely, the property that $[\omega]^{n-2} \circ \cdot : H^2_{dR}(M; \mathbb{R}) \to H^{2n-2}_{dR}(M; \mathbb{R})$ is an isomorphism, where $2n := \dim M$.

We study such a Lefschetz type property on almost-Kähler manifolds $(M, J, \omega, g)$ in relation to the existence of a cohomological decomposition of $H^2_{dR}(M; \mathbb{R})$. More precisely, we prove the following result.

**Theorem 2.3.** Let $(M, J, \omega, g)$ be a compact almost-Kähler manifold. Suppose that there exists a basis of $H^2_{dR}(X; \mathbb{R})$ represented by $g$-harmonic 2-forms which are of pure type with respect to $J$. Then the Lefschetz type property on 2-forms is satisfied.

Note that, by the hypothesis, it follows, in particular, that $J$ is $C^\infty$-pure-and-full and pure-and-full, [15 Theorem 3.7]. Note also that A. Fino and the second author provided in [15 several examples of compact non-Kähler solvmanifolds admitting a basis of harmonic representatives of pure-type with respect to the almost-complex structure. In [15 §2], T. Drăghici, T.-J. Li, and the third author ask whether such a Lefschetz type property on 2-forms is actually equivalent to $C^\infty$-fullness for every almost-Kähler nilmanifold and solvmanifold, without any further assumption; Theorem 2.3 and Proposition 4.1 provide results and examples in favour of a possibly positive answer to their question.

In [12 Theorem 1.1], starting with a compact complex surface $(M, J)$, it is shown that the dimension $h^2_J$ of the $J$-anti-invariant cohomology subgroup $H^2_J(M)$
of any **metric related** almost-complex structure \( \tilde{J} \) on \( M \) (namely, an almost-complex structure \( \tilde{J} \) on \( M \) inducing the same orientation as that one induced by \( J \) and with a common compatible metric), such that \( \tilde{J} \neq \pm J \), can be 0, 1, or 2, and a description of such almost-complex structures \( \tilde{J} \) having \( h_{\tilde{J}} \in \{1, 2\} \) is provided. Furthermore, it is conjectured that \( h_{\tilde{J}} = 0 \) for a generic almost-complex structure \( J \) on a compact 4-dimensional manifold, and that if \( h_{\tilde{J}} \geq 3 \), then \( J \) is integrable. [12, Conjecture 2.4, Conjecture 2.5]. One could set a similar question for higher dimensional manifolds, asking Question 5.2: are there examples of non-integrable almost-complex structures \( J \) on a compact 2\( n \)-dimensional manifold with \( h_{\tilde{J}} > n (n-1) \)?

Finally, we prove a Nomizu-type result for the subgroups \( H^\pm_{J}(M) \) of a completely-solvable solvmanifolds \( M = \Gamma \backslash G \) endowed with left-invariant almost-complex structures \( J \). More precisely, denote the Lie algebra associated to \( G \) by \( \mathfrak{g} \), and consider

\[
H^\pm_{J} = \left\{ \alpha = [\alpha] \in H^\bullet (\wedge^\ast \mathfrak{g}^*, \mathfrak{d}) : \alpha \in \wedge^\pm_{J} \otimes \mathfrak{g}^* \right\} \subseteq H^\pm_{dR}(M; \mathbb{R})
\]

the subgroup of \( H^\pm_{dR}(M; \mathbb{R}) \) that consists of classes admitting a left-invariant representative of type \((p, q) + (q, p)\), where \( \wedge^\pm_{J} \otimes \mathfrak{g}^* := \left( \wedge^{p+q} \otimes \mathfrak{g}^* \right)^* \cap \wedge^\ast \mathfrak{g}^* \); then the following result holds.

**Theorem 5.4.** Let \( M = \Gamma \backslash G \) be a solvmanifold endowed with a left-invariant almost-complex structure \( J \), and denote the Lie algebra naturally associated to \( G \) by \( \mathfrak{g} \). For any \( p, q \in \mathbb{N} \), the map \( j : H^\pm_{J} \rightarrow H^\pm_{dR}(M; \mathbb{R}) \) induced by left-translations is injective, and, if \( H^\pm_{dR}(M; \mathbb{R}) \) is a completely-solvable solvmanifold, then \( j : H^\pm_{J} \rightarrow H^\pm_{dR}(M; \mathbb{R}) \) is in fact an isomorphism.

In particular, it follows that \( \dim \mathbb{R} H^\pm_{J}(M) \leq n(n-1) \) for every left-invariant almost-complex structure on a completely-solvable solvmanifold.

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## 1. \( C^\infty \)-pure-and-full almost-complex structures

### 1.1. Subgroups of the de Rham cohomology of an almost-complex manifolds

We start by fixing some notation and recalling some recent results on cohomological properties of almost-complex manifolds; for more details see, e.g., \[20, 11, 12, 15, 11, 2, 13\], and the references therein.

Let \( J \) be a smooth almost-complex structure on a compact 2\( n \)-dimensional manifold \( M \). Denote by \( \wedge^\ast M \) the bundle of \( r \)-forms on \( M \); we denote with the same symbol \( \wedge^\ast M := \Gamma(M, \wedge^\ast M) \) the space of smooth global sections of the bundle \( \wedge^\ast M \). Then \( J \) extends to a complex automorphism of \( T^C M = TM \otimes \mathbb{C} \) such that \( T^C M = T^J_{1,0} M \oplus T^J_{0,1} M \), where \( T^J_{1,0} M \) and \( T^J_{0,1} M \) are the \((\pm i)\)-eigendoes. The action of \( J \) can be extended to the space \( \wedge^\ast (M; \mathbb{C}) \) of smooth global sections of the
bundle $\wedge^r(M; \mathbb{C}) := \wedge^r M \otimes \mathbb{C}$ getting the following decomposition:

$$\wedge^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \wedge_j^{p,q} M.$$  

Then the space $\wedge^r M$ of real smooth differential $r$-forms decomposes as

$$\wedge^r M = \bigoplus_{p+q=r, p \leq q} \wedge_j^{(p,q),(q,p)} (M)_\mathbb{R},$$

where, for $p < q$, (later on, we do not distinguish the cases $p < q$ and $p = q$)

$$\wedge_j^{(p,q),(q,p)} (M)_\mathbb{R} := \{ \alpha \in \wedge_j^{p,q} M \oplus \wedge_j^{q,p} M : \alpha = \overline{\alpha} \}, \quad \wedge_j^{(p,q)} (M)_\mathbb{R} := \{ \alpha \in \wedge_j^{p,q} M : \alpha = \overline{\alpha} \}.$$

In particular, for $r = 2$, we will adopt the following notation:

$$\wedge_j^{1,1}(M)_\mathbb{R} := \wedge_j^+ M, \quad \wedge_j^{(2,0),(0,2)} (M)_\mathbb{R} := \wedge_j^- M;$$

this is consistent with the decomposition in invariant and anti-invariant part of $\wedge^2 M$ under the natural action of $J$ on $\wedge^2 M$, given by $J \alpha(\cdot, \cdot) := \alpha(J \cdot, J \cdot)$.

We will refer to forms in $\wedge_j^{1,1}(M)_\mathbb{R}$, respectively $\wedge_j^{(2,0),(0,2)} (M)_\mathbb{R}$ as forms of pure type with respect to $J$.

For a finite set $S$ of pairs of integers, let

$$Z_j^S := \bigoplus_{(p,q) \in S, p \leq q} Z_j^{(p,q),(q,p)}, \quad B_j^S := \bigoplus_{(p,q) \in S, p \leq q} B_j^{(p,q),(q,p)},$$

where

$$Z_j^{(p,q),(q,p)} := \{ \alpha \in \wedge_j^{(p,q),(q,p)} (M)_\mathbb{R} : d \alpha = 0 \} ,$$

$$B_j^{(p,q),(q,p)} := \{ \beta \in \wedge_j^{(p,q),(q,p)} (M)_\mathbb{R} : \text{there exists } \gamma \text{ such that } d \gamma = \beta \} .$$

Define

$$H_j^S(M)_\mathbb{R} := \frac{Z_j^S}{B_j^S} .$$

Let $B$ be the space of d-exact forms. Since $Z_j^S = \frac{Z_j^S}{B_j^S} \otimes \mathbb{R}$, a natural inclusion

$$\rho_S : \frac{Z_j^S}{B_j^S} \to \frac{Z_j^S}{B_j^S}$$

is defined. As in [20], we will write $\rho_S \left( \frac{Z_j^S}{B_j^S} \right)$ simply as $\frac{Z_j^S}{B_j^S}$ and consequently the cohomology spaces $H_j^S(M)_\mathbb{R}$ can be identified as

$$H_j^S(M)_\mathbb{R} = \{ [\alpha] \in H_j^*(M; \mathbb{R}) : \alpha \in Z_j^S \} = \frac{Z_j^S}{B_j^S} .$$

Therefore, there is a natural inclusion

$$H_j^{1,1}(M)_\mathbb{R} \cap H_j^{(2,0),(0,2)}(M)_\mathbb{R} \subseteq H_j^2(M; \mathbb{R}) .$$

1.2. $C^\infty$-pure-and-full and pure-and-full almost-complex structures. As in [20], we set the following definition.

**Definition 1.1** ([20] Definition 2.2, Definition 2.3, Lemma 2.2). An almost-complex structure $J$ on a manifold $M$ is said to be

- $C^\infty$-pure if $H_j^{1,1}(M)_\mathbb{R} \cap H_j^{(2,0),(0,2)}(M)_\mathbb{R} = \{ 0 \}$,
- $C^\infty$-full if $H_j^{2}(M; \mathbb{R}) = H_j^{1,1}(M)_\mathbb{R} + H_j^{(2,0),(0,2)}(M)_\mathbb{R}$,
- $C^\infty$-pure-and-full if

$$H_j^2(M; \mathbb{R}) = H_j^{1,1}(M)_\mathbb{R} \oplus H_j^{(2,0),(0,2)}(M)_\mathbb{R} .$$
According to the previous notation, we will write

\[ H^+_J(M) := H^{(1,1)}_J(M)_{\mathbb{R}}, \quad H^-_J(M) := H^{(2,0),(0,2)}_J(M)_{\mathbb{R}}. \]

Similar definitions in terms of currents can be given, introducing the notion of pure-and-full almost-complex structure: we refer to \[20\] §2.2.2 for further details and results. More precisely, on an almost complex manifold \((M, J)\), the space \(\mathcal{E}_k(M)_{\mathbb{R}}\) of real \(k\)-currents has a decomposition \(\mathcal{E}_k(M)_{\mathbb{R}} = \bigoplus_{p+q=k} \mathcal{E}^j_{(p,q);(q,p)}(M)_{\mathbb{R}},\)

where \(\mathcal{E}^j_{(p,q);(q,p)}(M)_{\mathbb{R}}\) denotes the space of real \(k\)-currents of bi-dimension \((p, q) + (q, p)\).

Let \(\mathcal{Z}^j_{(2,0),(0,2)}\) and \(\mathcal{Z}^j_{(1,1)}\) denote the spaces of real d-closed currents of bi-dimension \((2, 0) + (0, 2)\), respectively \((1, 1)\), and \(\mathcal{B}^j_{(2,0),(0,2)}\) and \(\mathcal{B}^j_{(1,1)}\) denote the spaces of real d-exact currents of bi-dimension \((2, 0) + (0, 2)\), respectively \((1, 1)\). Denote by \(\mathcal{B}\) the space of boundaries. Let, as in \[20\],

\[ H^j_{(1,1)}(M)_{\mathbb{R}} := \left\{ \alpha \in H_2(M; \mathbb{R}) : \alpha \in \mathcal{Z}^j_{(1,1)} \right\} = \frac{\mathcal{Z}^j_{(1,1)}}{\mathcal{B}}, \]

\[ H^j_{(2,0),(0,2)}(M)_{\mathbb{R}} := \left\{ \alpha \in H_2(M; \mathbb{R}) : \alpha \in \mathcal{Z}^j_{(2,0),(0,2)} \right\} = \frac{\mathcal{Z}^j_{(2,0),(0,2)}}{\mathcal{B}}. \]

We recall the following definition.

**Definition 1.2** (\[20\] Definition 2.15, Definition 2.16). An almost complex structure \(J\) on a manifold \(M\) is said to be pure if \(H^j_{(1,1)}(M)_{\mathbb{R}} \cap H^j_{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\}\). It is said to be full if \(H_2(M; \mathbb{R}) = H^j_{(1,1)}(M)_{\mathbb{R}} + H^j_{(2,0),(0,2)}(M)_{\mathbb{R}}\). Therefore, an almost complex structure \(J\) is pure-and-full if and only if \(H_2(M; \mathbb{R}) = H^j_{(1,1)}(M)_{\mathbb{R}} \oplus H^j_{(2,0),(0,2)}(M)_{\mathbb{R}}\).

In \[20\] Proposition 2.1] it is shown that, given a compact complex manifold \((M, J)\) of complex dimension \(n\), if \(n = 2\) or \(J\) is Kähler, then \(J\) is \(C^\infty\)-pure-and-full, and \(H^j_{(1,1)}(M)_{\mathbb{R}} \simeq H^j_{\partial}(M)\cap H^j_{\bar{\partial}}(M)_{\mathbb{R}}\) and \(H^j_{(2,0),(0,2)}(M)_{\mathbb{R}} \simeq \left( H^j_{\partial}(M) \oplus H^j_{\bar{\partial}}(M) \right) \cap H^j_{\partial\bar{\partial}}(M).\) In view of this result, the subgroups \(H^j_{(1,1)}(M)_{\mathbb{R}}\) and \(H^j_{(2,0),(0,2)}(M)_{\mathbb{R}}\) of the de Rham cohomology can be viewed as an analogue of the Dolbeault cohomology groups for non-integrable almost-complex structures.

In \[11\] Theorem 2.3] it is proven the following result.

**Theorem 1.3** (\[11\] Theorem 2.3]). If \(M\) is a compact manifold of dimension 4, then any almost-complex structure \(J\) on \(M\) is \(C^\infty\)-pure-and-full.

This is no more true in dimension higher than 4: in \[13\] Example 3.3, a compact non-\(C^\infty\)-pure-almost-complex structure on a 6-dimensional nilmanifold is constructed. Therefore, the previous theorem can be considered a sort of Hodge decomposition theorem in the non-Kähler case.

2. **Cohomological properties of almost-Kähler manifolds**

2.1. **Lefschetz type property on almost-Kähler manifolds with pure-type harmonic representatives.** Given a compact \(2n\)-dimensional almost-Kähler manifold \((M, J, \omega, g)\), we are interested in studying the property of being \(C^\infty\)-pure-and-full.

Firstly we recall the following result.
Proposition 2.1 ([11, Proposition 2.8], [15, Proposition 3.2]). If $J$ is an almost-complex structure on a compact manifold $M$ and $J$ admits a compatible symplectic structure, then $J$ is $C^\infty$-pure.

Furthermore, A. Fino and the second author proved that an almost-Kähler manifold admitting a basis of harmonic 2-forms whose elements are of pure type with respect to the almost-complex structure is $C^\infty$-pure-and-full and pure-and-full, [15, Theorem 3.7]; they also provided several examples of compact non-Kähler solvmanifolds satisfying the above assumption in [15].

To the purpose of studying the property of being $C^\infty$-pure-and-full on almost-Kähler manifolds, we recall the following definition.

Definition 2.2. Given a compact $2n$-dimensional symplectic manifold $(M, \omega)$, denote by

$$L_\omega : \wedge^2 M \to \wedge^{2n-2} M, \quad L_\omega(\alpha) := \omega^{n-2} \wedge \alpha,$$

the Lefschetz type operator (on 2-forms) associated with $\omega$.

Then one says that the compact $2n$-dimensional almost-Kähler manifold $(M, J, \omega, g)$ satisfies the Lefschetz type property (on 2-forms) if $L_\omega$ takes $g$-harmonic 2-forms to $g$-harmonic $(2n-2)$-forms.

Furthermore, we recall some notions and results from [6, 22, 27], see also [23, 7]. Let $(M, \omega)$ be a compact $2n$-dimensional symplectic manifold. Extend $\omega^{-1} : T^*M \to TM$ to the whole exterior algebra of $T^*M$. For any $k \in \mathbb{N}$, the symplectic $\star_\omega$ operator is defined as

$$\star_\omega : \wedge^k M \to \wedge^{2n-k} M, \quad \beta \wedge \star_\omega \alpha = \omega^{-1}(\alpha, \beta) \frac{\omega^n}{n!}, \quad \forall \alpha, \beta \in \wedge^k M.$$

One can prove that $\star_\omega^2 = \text{id}_{\wedge^\cdot M}$, [6, Lemma 2.1.2].

For any $k \in \mathbb{N}$, define the symplectic co-differential operator

$$\delta_\omega : \wedge^k M \to \wedge^{k-1} M, \quad \delta_\omega|_{\wedge^k M} := (-1)^{k+1} \star_\omega d \star_\omega;$$

this operator has been studied by J.-L. Brylinski in [6] for Poisson manifolds; in the context of generalized complex geometry, see, e.g., [16], it can be interpreted as the symplectic counterpart of the operator $d^c := -i(\partial - \bar{\partial})$ in complex geometry, see [7].

By definition, $(M, \omega)$ satisfies the Hard Lefschetz Condition if, for each $k \in \mathbb{N}$, the map

$$[\omega]^k \circ : H^{n-k}_{dR}(M; \mathbb{R}) \to H^{n+k}_{dR}(M; \mathbb{R})$$

is an isomorphism. O. Mathieu, [22, Corollary 2], and, independently, D. Yan, [27, Theorem 0.1], proved that, given a compact symplectic manifold $(M, \omega)$, any de Rham cohomology class has a (possibly non-unique) $\omega$-symplectically harmonic representative (that is, a $d$-closed $\delta_\omega$-closed representative) if and only if the Hard Lefschetz Condition holds.

We can now prove the following result.

Theorem 2.3. Let $(M, J, \omega, g)$ be a compact almost-Kähler manifold. Suppose that there exists a basis of $H^2_{dR}(X; \mathbb{R})$ represented by $g$-harmonic 2-forms which are of pure type with respect to $J$. Then the Lefschetz type property on 2-forms is satisfied.
Proof. Recall that, on a $2n$-dimensional almost-Kähler manifold $(M, J, \omega, g)$, the Hodge $*_{g}$ operator and the symplectic $*_{\omega}$ operator are related by $*_{\omega} = *_{g} J$ [6 Theorem 2.4.1, Remark 2.4.4]. Therefore, for forms of pure type with respect to $J$, the properties of being $g$-harmonic and of being $\omega$-symplectically harmonic are equivalent. The theorem follows noting that, [27] Lemma 1.2], $[\mathcal{L}_{\omega}, d] = 0$ and $[\mathcal{L}_{\omega}, \delta_{\omega}] = d$, hence $\mathcal{L}_{\omega}$ sends $\omega$-symplectically harmonic 2-forms (of pure type with respect to $J$) to $\omega$-symplectically harmonic $(2n-2)$-forms (of pure type with respect to $J$).

\[ \square \]

Remark 2.4. We note that if $(M, J, \omega, g)$ is a compact $2n$-dimensional almost-Kähler manifold satisfying the Lefschetz type property on 2-forms and $J$ is $C^\infty$-full, then $J$ is $C^\infty$-pure-and-full and pure-and-full.

Indeed, we have already remarked that $J$ is $C^\infty$-pure, see Proposition 2.1. Moreover, since $J$ is $C^\infty$-full, $J$ is also pure by [20] Proposition 2.5. We recall now the argument in [15] to prove that $J$ is also full.

Firstly, note that if the Lefschetz type property on 2-forms holds, then $[\omega^{n-2}] \sim \cdots H^{2n-2}_{dR}(M; \mathbb{R}) \rightarrow H^{2n-2}_{dR}(M; \mathbb{R})$ is an isomorphism. Therefore, we get that

\[ H^{2n-2}_{dR}(M; \mathbb{R}) = H^{(n,n-2),(n-2,n)}(M)_{\mathbb{R}} + H^{(n-1,n-1)}(M)_{\mathbb{R}} \]

indeed, (following the argument in [15] Theorem 4.1]), since $[\omega^{n-2}] \sim \cdots H^{2}_{dR}(M; \mathbb{R}) \rightarrow H^{2n-2}_{dR}(M; \mathbb{R})$ is in particular surjective, we have

\[ H^{2n-2}_{dR}(M; \mathbb{R}) = [\omega^{n-2}] \sim H^{2}_{dR}(M; \mathbb{R}) = [\omega^{n-2}] \sim \left( H^{(2,0),(0,2)}(M)_{\mathbb{R}} \oplus H^{(1,1)}(M)_{\mathbb{R}} \right) \]

yielding the above decomposition of $H^{2n-2}_{dR}(M; \mathbb{R})$. Then, it follows that $J$ is also full, see, for example, [1] Theorem 2.1.

2.2. A family of almost-Kähler manifolds satisfying the Lefschetz type property on 2-forms. Let $n$ be the 6-dimensional nilpotent Lie algebra whose structure equations, with respect to a basis $\{e^j\}_{j \in \{1, \ldots, 6\}}$ of $n^*$, are given by

\[ d e^1 = d e^2 = d e^3 = 0, \quad d e^4 = e^{23}, \quad d e^5 = e^{13}, \quad d e^6 = e^{12} \]

(where we write $e^{jk}$ instead of $e^j \wedge e^k$). Using a result by Mal’tsev, [21] Theorem 7], the connected simply-connected Lie group $G$ associated with $n$ admits a discrete co-compact subgroup $\Gamma$: let $N := \Gamma \backslash G$ be the (compact) nilmanifold obtained as a quotient of $G$ by $\Gamma$. Note that $N$ is not formal by a theorem of K. Hasegawa's, [17] Theorem 1, Corollary].

Fix $\alpha > 1$ and take

\[ \omega_\alpha := e^{14} + \alpha \cdot e^{25} + (\alpha - 1) \cdot e^{36}; \]

since $d \omega_\alpha = 0$ and $\omega_\alpha^3 \neq 0$, we get that $\omega_\alpha$ is a left-invariant symplectic form on $N$. Set

\[ J_\alpha e_1 := e_4, \quad J_\alpha e_2 := \alpha e_5, \quad J_\alpha e_3 := (\alpha - 1) e_6, \]

\[ J_\alpha e_4 := -e_1, \quad J_\alpha e_5 := -\frac{1}{\alpha} e_2, \quad J_\alpha e_6 := -\frac{1}{\alpha - 1} e_3, \]

where $\{e_1, \ldots, e_6\}$ denotes the global dual frame of $\{e^1, \ldots, e^6\}$ on $N$. It is immediate to check that
• setting \( g_\alpha(\cdot, \cdot) := \omega_\alpha(\cdot, J_\alpha) \), the triple \((J_\alpha, \omega_\alpha, g_\alpha)\) gives rise to a family of left-invariant almost-Kähler structures on \(N\);

• denoting by

\[
E^{1}_\alpha := e^{1}, \quad E^{2}_\alpha := \alpha \ e^{2}, \quad E^{3}_\alpha := (\alpha - 1) \ e^{3}, \quad E^{4}_\alpha := e^{4}, \quad E^{5}_\alpha := e^{5}, \quad E^{6}_\alpha := e^{6},
\]

then \( \{E^{1}_\alpha, \ldots, E^{6}_\alpha\} \) is a \( g_\alpha \)-orthonormal co-frame on \(N\); with respect to this new co-frame, we easily obtain the following structure equations:

\[
d E^{1}_\alpha = d E^{2}_\alpha = d E^{3}_\alpha = 0, \quad d E^{4}_\alpha = \frac{1}{\alpha(\alpha - 1)} E^{23}_\alpha, \quad d E^{5}_\alpha = \frac{1}{\alpha - 1} E^{13}_\alpha, \quad d E^{6}_\alpha = \frac{1}{\alpha} E^{12}_\alpha.
\]

Then,

\[
\varphi^{1}_\alpha := E^{1}_\alpha + i E^{2}_\alpha, \quad \varphi^{2}_\alpha := E^{2}_\alpha + i E^{5}_\alpha, \quad \varphi^{3}_\alpha := E^{3}_\alpha + i E^{6}_\alpha,
\]

are \((1,0)\)-forms with respect to the almost-complex structure \(J_\alpha\), and

\[
\omega_\alpha = E^{14}_\alpha + E^{25}_\alpha + E^{36}_\alpha.
\]

By a result of K. Nomizu’s, \cite{Nomizu:1953} Theorem 1, see Theorem \ref{thm:om} the de Rham cohomology of \(N\) is straightforwardly computed:

\[
H^{2}_{dR}(N; \mathbb{R}) \cong \mathbb{R} \left( E^{15}_\alpha, E^{16}_\alpha, E^{24}_\alpha, E^{26}_\alpha, E^{34}_\alpha, E^{35}_\alpha, E^{14}_\alpha + \frac{1}{\alpha} E^{25}_\alpha, \frac{1}{\alpha - 1} E^{25}_\alpha + \frac{1}{\alpha} E^{36}_\alpha \right)
\]

(where we have listed the \( g_\alpha \)-harmonic representatives instead of their classes).

Note that the listed \( g_\alpha \)-harmonic representatives of \( H^{2}_{dR}(N; \mathbb{R}) \) are of pure type with respect to \(J_\alpha\); hence, the almost-complex structure \(J_\alpha\) is \( \mathcal{C}^\infty \)-pure-and-full by \cite{Liu:2004} Theorem 3.7; in particular, note that

\[
H^{2}_{dR}(N; \mathbb{R}) \cong \mathbb{R} \left( i \alpha \varphi^{11}_\alpha + i \varphi^{22}_\alpha, i (\alpha - 1) \varphi^{22}_\alpha + i \alpha \varphi^{33}_\alpha, \Im \varphi^{12}_\alpha, \Im \varphi^{13}_\alpha, \Im \varphi^{32}_\alpha \right) \oplus \left( \Im \varphi^{12}_\alpha, \Im \varphi^{13}_\alpha, \Im \varphi^{32}_\alpha \right),
\]

hence \( h^{1}_{dR}(N) = 5 \) and \( h^{1}_{dR}(N) = 3 \).

Moreover, one explicitly notes that

\[
\mathcal{L}_{\omega_\alpha} E^{15}_\alpha = E^{1536}_\alpha = * g_\alpha E^{24}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{16}_\alpha = E^{1625}_\alpha = * g_\alpha E^{34}_\alpha,
\]

\[
\mathcal{L}_{\omega_\alpha} E^{24}_\alpha = E^{2436}_\alpha = * g_\alpha E^{15}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{26}_\alpha = E^{2614}_\alpha = * g_\alpha E^{35}_\alpha,
\]

\[
\mathcal{L}_{\omega_\alpha} E^{34}_\alpha = E^{3425}_\alpha = * g_\alpha E^{16}_\alpha, \quad \mathcal{L}_{\omega_\alpha} E^{35}_\alpha = E^{3514}_\alpha = * g_\alpha E^{26}_\alpha,
\]

while

\[
\mathcal{L}_{\omega_\alpha} \left( E^{14}_\alpha + \frac{1}{\alpha} E^{25}_\alpha \right) = - \frac{\alpha + 1}{\alpha} E^{1245}_\alpha - \frac{1}{\alpha} E^{2356}_\alpha - E^{1346}_\alpha
\]

where

\[
d * g_\alpha \mathcal{L}_{\omega_\alpha} \left( E^{14}_\alpha + \frac{1}{\alpha} E^{25}_\alpha \right) = d \left( - \frac{\alpha + 1}{\alpha} E^{1245}_\alpha - E^{2356}_\alpha - \frac{1}{\alpha} E^{14}_\alpha \right) = 0,
\]

and, by a similar computation, \( d * g_\alpha \mathcal{L}_{\omega_\alpha} (e^{25} + e^{36}) = 0 \). This proves explicitly that \( \omega_\alpha \) satisfies the Lefschetz type property on 2-forms.

The nilmanifold \( N \) is not formal by a theorem of K. Hasegawa’s, \cite{Has:1974} Theorem 1, Corollary. The non-formality of \( M \) can be also proved by giving a non-zero triple Massey product on \(N\), see \cite{Massey:1977}: since

\[
[E^{1}_\alpha] \cup [E^{3}_\alpha] = (\alpha - 1) [d E^{3}_\alpha] = 0, \quad [E^{3}_\alpha] \cup [E^{2}_\alpha] = -\alpha (\alpha - 1) [d E^{2}_\alpha] = 0,
\]
we get that the triple Massey product
\[ \langle [E_1^6], [E_3^3], [E_5^2] \rangle = -(\alpha - 1) [E_6^{25} + \alpha E_6^{14}] \]
does not vanish, and hence \( N \) is not formal.

In summary, we have proven the following result.

**Proposition 2.5.** There is a non-formal 6-dimensional nilmanifold \( N \) endowed with a 1-parameter family \( \{ (J_\alpha, \omega_\alpha, g_\alpha) \}_{\alpha > 1} \) of left-invariant almost-Kähler structures being \( C^\infty \)-pure-and-full and pure-and-full and satisfying the Lefschetz type property on 2-forms.

**Remark 2.6.** It has to be noted that \( \omega_\alpha \wedge : \wedge^2 \mathbb{R}^6 \to \wedge^4 \mathbb{R}^6 \) induces an isomorphism in cohomology \( [\omega_\alpha] \sim : H^{2}_{dR}(\mathbb{R}, \mathbb{R}) \to H^{4}_{dR}(\mathbb{R}, \mathbb{R}) \), while, accordingly to [5 Theorem A], \( [\omega_\alpha]^2 \sim : H^{2}_{dR}(\mathbb{R}, \mathbb{R}) \to H^{4}_{dR}(\mathbb{R}, \mathbb{R}) \) is not an isomorphism.

### 3. Almost-Kähler \( C^\infty \)-pure-and-full structures

#### 3.1. The Nakamura manifold of completely solvable type.

Take \( A \in \text{SL}(2; \mathbb{Z}) \) with two different real eigenvalues \( e^\lambda \) and \( e^{-\lambda} \) with \( \lambda > 0 \), and fix \( P \in \text{GL}(2; \mathbb{R}) \) such that \( PAP^{-1} = \text{diag}(e^\lambda, e^{-\lambda}) \). For example, take
\[
A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P := \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix}
\]
and consequently \( \lambda = \log \frac{1+\sqrt{5}}{2} \). Let \( M^6 := M^6(\lambda) \) be the compact complex manifold
\[
M^6 := S^1 \times \frac{\mathbb{R} \times T^2_C \times \mathbb{R}^2 \times \mathbb{R}^2}{T_1}
\]
where \( T^2_C \) is the 2-dimensional complex torus \( T^2_C := \mathbb{C}/T_{2\pi i} \), and \( T_1 \) acts on \( \mathbb{R} \times T^2_C \) as \( T_1 (x^1, x^3, x^4, x^5, x^6) := (x^1 + \lambda, e^{-\lambda} x^3, e^{\lambda} x^4, e^{-\lambda} x^5, e^{\lambda} x^6) \). The manifold \( M^6 \) can be seen as a compact quotient of a completely-solvable Lie group by a discrete co-compact subgroup, [14 Example 3.1]; (denote the Lie algebra naturally associated to the completely-solvable Lie group of \( M^6 \) by \( \mathfrak{g} \)). Using coordinates \( x^2 \) on \( S^1 \), \( x^1 \) on \( \mathbb{R} \) and \( (x^3, x^4, x^5, x^6) \) on \( T^2_C \), we set
\[
e^1 := dx^1, \quad e^2 := dx^2, \quad e^3 := e^{x^1} dx^3, \quad e^4 := e^{-x^1} dx^4, \quad e^5 := e^{x^1} dx^5, \quad e^6 := e^{-x^1} dx^6
\]
as a basis for \( \mathfrak{g}^* \); therefore, with respect to \( \{ e^i \}_{i \in \{1, \ldots, 6\}} \), the structure equations are the following:
\[
d e^1 = d e^2 = 0, \quad d e^3 = e^{13}, \quad d e^4 = -e^{14}, \quad d e^5 = e^{15}, \quad d e^6 = -e^{16}.
\]

#### 3.2. The de Rham cohomology of the Nakamura manifold.

Let \( J \) be the almost-complex structure on \( M^6 \) defined by the complex \((1,0)\)-forms given by
\[
\varphi^1 := \frac{1}{2} (e^1 + i e^2), \quad \varphi^2 := e^3 + i e^5, \quad \varphi^3 := e^4 + i e^6.
\]

It is straightforward to check that \( J \) is integrable. Being \( M^6 \) a compact quotient of a completely-solvable Lie group, one computes the
de Rham cohomology of $M^6$ easily by A. Hattori’s theorem [19, Corollary 4.2], see Theorem 5.3.

$$H^1_{dR}(M^6; \mathbb{C}) \simeq \mathbb{C} \langle \varphi^1, \bar{\varphi}^1 \rangle, \quad H^2_{dR}(M^6; \mathbb{C}) \simeq \mathbb{C} \langle \varphi^{12}, \varphi^{32}, \varphi^{23}, \varphi^{33} \rangle,$$

$$H^3_{dR}(M^6; \mathbb{C}) \simeq \mathbb{C} \langle \varphi^{132}, \varphi^{132}, \varphi^{123}, \varphi^{233}, \varphi^{331}, \varphi^{231}, \varphi^{123} \rangle$$

(for the sake of clearness, we write, for example, $\varphi^{AB}$ in place of $\varphi^A \wedge \varphi^B$ and we list the harmonic representatives with respect to the metric $g := \sum_{j=1}^{3} \varphi^j \otimes \bar{\varphi}^j$ instead of their classes). Therefore, $M^6$ is geometrically formal, i.e., the product of $g$-harmonic forms is still $g$-harmonic, and therefore it is formal, namely the de Rham complex of $M$ is formal as a differentially graded algebra, see, e.g., [9]. Furthermore, it can be easily checked that

$$\omega := e^{12} + e^{34} + e^{56}$$

gives rise to a symplectic structure on $M^6$ satisfying the Hard Lefschetz Condition. We obtain the following result.

**Proposition 3.1 (14, Proposition 3.2).** The manifold $M^6$ is formal and it admits a symplectic form $\omega$ satisfying the Hard Lefschetz Condition.

Note also that $\hat{\omega} := \frac{1}{2} (\varphi^{11} + \varphi^{22} + \varphi^{33})$ is not d-closed but $d\hat{\omega}^2 = 0$, from which it follows that the manifold $M^6$ admits a balanced metric.

Moreover, since $M^6$ is a compact quotient of a completely-solvable Lie group, by the K. Hasegawa’s theorem [18, Main Theorem], we have the following result, see also [14, Theorem 3.3]. (We recall that a compact complex manifold is said to belong to class $C$ of Fujiki if it admits a proper modification from a Kähler manifold.)

**Theorem 3.2 (18, Main Theorem).** The manifold $M^6$ admits no Kähler structure and it is not in class $C$ of Fujiki.

3.3. **An almost-Kähler structure on the Nakamura manifold.** By K. Hasegawa’s theorem [18, Main Theorem], any integrable complex structure on $M^6$ (for example, the $J$ defined in §3.2) does not admit any symplectic structure compatible with it. Therefore, we consider the almost-complex structure $J'$ defined by

$$J' e^1 := -e^2, \quad J' e^3 := -e^4, \quad J' e^5 := -e^6;$$

considering

$$\psi^1 := \frac{1}{2} (e^1 + ie^2), \quad \psi^2 := e^3 + ie^4, \quad \psi^3 := e^5 + ie^6$$

as a co-frame for the space of $(1,0)$-forms on $(M^6, J')$, one can compute

$$d\psi^1 = 0, \quad d\psi^2 = \psi^{12} + \bar{\psi}^{12}, \quad d\psi^3 = \psi^{13} + \bar{\psi}^{13},$$

from which it is clear that $J'$ is not integrable. Note that the $J'$-compatible 2-form

$$\omega' := e^{12} + e^{34} + e^{56}$$

is d-closed. Hence, $(M^6, J', \omega')$ is an almost-Kähler manifold. Moreover, recall that

$$H^2_{dR}(M^6, \mathbb{R}) \simeq \mathbb{R} \langle i \psi^{11}, i \psi^{22}, i \psi^{33}, i (\psi^{23} + \bar{\psi}^{32}) \rangle \oplus \mathbb{R} \langle i (\psi^{23} - \bar{\psi}^{32}) \rangle \subseteq H^2_{j'}(M^6, \mathbb{R}) \subseteq H^2_{j'}(M^6, \mathbb{R})$$
where we have listed the harmonic representatives with respect to the metric \( g' := \sum_{j=1}^{6} e^{j} \circ e^{j} \) instead of their classes; note that the listed \( g' \)-harmonic representatives are of pure type with respect to \( J' \). Therefore, \( J' \) is obviously \( C^\infty \)-full; it is also \( C^\infty \)-pure by \cite[Proposition 3.2]{15} or \cite[Proposition 2.8]{11}, see Proposition 2.1. Moreover, let \( \sum_{j} \).

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Set
\[ v_1 := e^{15} - e^{26}, \quad v_2 := e^{16} + e^{25}, \quad v_3 := e^{35} - e^{46}, \quad v_4 := e^{36} + e^{45}, \]
\[ v_5 := e^{13} + e^{24}, \quad v_6 := e^{23} - e^{14}, \quad v_7 := e^{12}, \quad v_8 := e^{34}. \]

Consider the almost-Kähler structure \((J, \omega, g)\) on \(X\) defined by
\[ Je^1 := -e^6, \quad Je^2 := -e^5, \quad Je^3 := -e^4, \quad \omega := e^{16} + e^{25} + e^{34}. \]

We easily get that
\[ \mathbb{R} \langle v_2, v_3 + v_5, v_4 - v_6, v_8 \rangle \subseteq H^+_J(X), \quad \mathbb{R} \langle v_1, v_3 - v_5, v_4 + v_6 \rangle \subseteq H^+_J(X). \]

We claim that the previous inclusions are actually equalities, and in particular that \(J\) is a non-C\(^\infty\)-full almost-Kähler structure on \(X\).

Indeed, we firstly note that, by [15, Proposition 3.2] or [11, Proposition 2.8], see Proposition 2.1, \(J\) is a non-C\(^\infty\)-full almost-Kähler structure.

We claim that the previous inclusions are actually equalities, and in particular that \(J\) is a non-C\(^\infty\)-full almost-Kähler structure on \(X\).

Furthermore, the Lefschetz type operator of the almost-Kähler structure
\(H^+_J(X)\) is not full.

Let \(L\) be the Lefschetz type operator of the almost-Kähler structure \((J, \omega, g)\). Then, we have \(L(e^{12}) = e^{1234} = d(e^{243})\), i.e., \(L\) does not take \(g\)-harmonic 2-forms in \(g\)-harmonic 4-forms.

Hence, we have proved the following result.

**Proposition 4.1.** Let \(X := \mathbb{Z}[\mathbb{C}^3] \setminus (\mathbb{C}^3, *)\) be the real manifold underlying the Iwasawa manifold. Then there exists an almost-Kähler structure \((J, \omega, g)\) on \(X\) which is C\(^\infty\)-pure and non-C\(^\infty\)-full. Furthermore, the Lefschetz type operator of the almost-Kähler structure \((J, \omega, g)\) does not take \(g\)-harmonic 2-forms to \(g\)-harmonic 4-forms.

5. **Almost-complex manifolds with large anti-invariant cohomology**

Given an almost-complex structure \(J\) on a compact manifold \(M\), it is natural to ask how large the cohomology subgroup \(H^J_{(2,0),(0,2)}(M)\) can be. In this direction, T. Drăghici, T.-J. Li, and the third author raised the following question in [12].

**Question 5.1.** ([12, Conjecture 2.5]). Are there compact 4-dimensional manifold \(M\) endowed with non-integrable almost-complex structures \(J\) such that \(\dim_{\mathbb{R}} H^J_{(2,0)}(M) \geq 3\)?
We present here a 1-parameter family \( \{ J_t \} \) of (non-integrable) almost-complex structures on the 6-dimensional torus \( T^6 \) having \( h_{J_t} := \dim_{\mathbb{R}} H^2_{J_t}(T^6; \mathbb{R}) \) greater than 3, see also [11, §4]. For \( t \) small enough, set \( \alpha_t := : \alpha_t(x^3) \in C^\infty (T^6) \) such that \( \alpha_0(x^3) = 1 \) and set

\[
\varphi^1_t := d x^1 + i \alpha_t d x^4, \quad \varphi^2_t := d x^2 + i d x^5, \quad \varphi^3_t := d x^3 + i d x^6;
\]

therefore, the structure equations are

\[
d \varphi^1_t = i d \alpha_t \wedge d x^4, \quad d \varphi^2_t = 0, \quad d \varphi^3_t = 0.
\]

Straightforward computations give that the \( J \)-anti-invariant \( d \)-closed 2-forms are of the type

\[
\psi = \frac{C}{\alpha_t} (d x^{13} - \alpha_t d x^{46}) + D (d x^{16} - \alpha_t d x^{34}) + E (d x^{23} - d x^{56}) + F (d x^{26} - d x^{35}),
\]

where \( C, D, E, F \in \mathbb{R} \) (we shorten \( d x^j \wedge d x^k \) by \( d x^{jk} \)). Moreover, the forms \( d x^{23} - d x^{56} \) and \( d x^{26} - d x^{35} \) are clearly harmonic with respect to the standard flat metric \( \sum_{j=1}^6 d x^j \otimes d x^j \), while the classes of \( d x^{16} - \alpha_t d x^{34} \) and \( d x^{13} - \alpha_t d x^{46} \) are non-zero, their harmonic parts being non-zero. Hence, we get that \( h_{\tilde{J}_t} = 4 \) and

\[
h_{J_t} = 4 \quad \text{for small} \quad t \neq 0.
\]

In the general case, we ask the following natural question.

**Question 5.2.** Are there examples of non-integrable almost-complex structures \( J \) on a compact \( 2n \)-dimensional manifold with \( \dim_{\mathbb{R}} H^2_{J}(M) > n(n-1) \)?

Consider now a solvmanifold \( M = \Gamma \backslash G \), namely, a compact quotient of a connected simply-connected solvable Lie group \( G \) by a co-compact discrete subgroup \( \Gamma \). Denote the Lie algebra naturally associated to \( G \) by \( g \), and consider \( (\Lambda^* g^*, d) \) the subcomplex of the de Rham complex \((\Lambda^* M, d)\) given by the left-invariant differential forms. The following result by K. Nomizu [25] and A. Hattori [19] holds.

**Theorem 5.3** ([25, Theorem 1], [19, Theorem 4.2]). Let \( M \) be a nilmanifold or, more in general, a completely-solvable solvmanifold. Then \( H^*(\Lambda^* g^*, d) \simeq H^*_{dr}(M; \mathbb{R}) \).

Let \( J \) be a left-invariant almost-complex structure on \( M \), namely, an almost-complex structure on \( M \) induced by an almost-complex structure on \( G \) that is invariant under the action of \( G \) on itself given by left-translations. Given \( p, q \in \mathbb{N} \), denote by

\[
H^*_j(p,q) (\Lambda^* g^*, (p,q) (\Lambda^* g^*), (p,q) (\Lambda^* g^*)) \subseteq H^*_{dr}(M; \mathbb{R})
\]

the subgroup (see, e.g., [8, Lemma 9]) of \( H^*_{dr}(M; \mathbb{R}) \) that consists of classes admitting a left-invariant representative of type \( (p,q) + (q,p) \), where \( \Lambda^j_{(p,q)} (\Lambda^* g^*)^* := (\Lambda^j_{(p,q)} (\Lambda^* g^*))^* \cap \Lambda^* g^* \).

Using Belgun’s symmetrization trick, [4, Theorem 7], one can prove the following Nomizu-type result, which relates the subgroups \( H^*_j(p,q) (\Lambda^* g^*) (M; \mathbb{R}) \) with their left-invariant part \( H^*_j(p,q) (\Lambda^* g^*) (\mathbb{R}) \).
Theorem 5.4. Let $M = \Gamma \backslash G$ be a solvmanifold endowed with a left-invariant almost-complex structure $J$, and denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$. For any $p, q \in \mathbb{N}$, the map

$$j: H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R} \to H^p_j\{(p,q)\}(M)_\mathbb{R}$$

induced by left-translations is injective, and, if $H^p_{\sigma R}(\wedge^n \mathfrak{g}^*, d) \simeq H^p_{\sigma R}(M; \mathbb{R})$ (for instance, if $M$ is a completely-solvable solvmanifold), then $j: H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R} \to H^p_j\{(p,q)\}(M)_\mathbb{R}$ is in fact an isomorphism.

Proof. Since $J$ is left-invariant, left-translations induce the map $j: H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R} \to H^p_j\{(p,q)\}(M)_\mathbb{R}$.

Since, by J. Milnor’s Lemma [24, Lemma 6.2], $G$ is unimodular, one can take in particular a bi-invariant volume form $\eta$ on $M$ such that $\int_M \eta = 1$. Consider the F. A. Belgun symmetrization map in [4, Theorem 7], namely,

$$\mu: \wedge \mathfrak{g} \to \wedge \mathfrak{g}^*, \mu(\alpha) := \int_M \alpha|_m \eta(m).$$

Since $\mu$ commutes with $d$ by [4, Theorem 7], it induces the map $\mu: H^p_{\sigma R}(M; \mathbb{R}) \to H^p\{(\wedge^n \mathfrak{g}^*, d)\}$, and, since $\mu$ commutes with $J$, it preserves the bi-graduation; therefore it induces the map $\mu: H^p_j\{(p,q)\}(M) \to H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R}$. Moreover, since $\mu$ is the identity on the space of left-invariant forms by [4, Theorem 7], we get the commutative diagram

$$H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R} \xrightarrow{j} H^p_j\{(p,q)\}(M)_\mathbb{R} \xrightarrow{\mu} H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R}$$

hence $j: H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R} \to H^p_j\{(p,q)\}(M)_\mathbb{R}$ is injective, and $\mu: H^p_j\{(p,q)\}(M)_\mathbb{R} \to H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R}$ is surjective.

Furthermore, when $H^p(\wedge^n \mathfrak{g}^*, d) \simeq H^p_{\sigma R}(M; \mathbb{R})$ (for instance, when $M$ is a completely-solvable solvmanifold, by A. Hattori’s theorem [19, Theorem 4.2], see Theorem [5.9]), since $\mu|_{\wedge^n \mathfrak{g}^*} = \text{id}|_{\wedge^n \mathfrak{g}^*}$ by [4, Theorem 7], we get that $\mu: H^p_j\{(p,q)\}(M)_\mathbb{R} \to H^p\{(\wedge^n \mathfrak{g}^*, d)\}$ is the identity map, and hence $\mu: H^p_j\{(p,q)\}(M)_\mathbb{R} \to H^p_j\{(p,q)\}(\mathfrak{g})_\mathbb{R}$ is also injective, and hence an isomorphism.

In particular, if $M = \Gamma \backslash G$ is a 2n-dimensional completely-solvable solvmanifold endowed with a left-invariant almost-complex structure $J$, then

$$\dim \mathbb{R} H_j^-(M) \leq n(n-1) \quad \text{and} \quad \dim \mathbb{R} H^+_j(M) \leq n^2;$$

this provides a partial negative answer to Question [5.2].

References

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