Università di Pisa

Dipartimento di Informatica

Technical Report : TR-96-34

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October 25, 1996

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Parsing MELL Proof Nets

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We propose a new formulation for full (weakening and constants included) multiplicative and exponential (MELL) proof nets, allowing a complete set of rewriting rules to parse them. The recognizing grammar defined by such a rewriting system (confluent and strong normalizing on the new proof nets) gives a correctness criterion that we show equivalent to the Danos-Regnier one.

1 Introduction

Before the arrival on the scene of linear logic there were essentially two possible formulations for proofs: *sequent calculus* and *natural deduction*. Both enjoying the property that each application of a rule is *correct* (locally correct in the case of sequent calculus and globally correct in the case of natural deduction). Namely, each instance of a rule of the calculus transforms a (correct) proof into another (correct) proof. It was a general belief that any reasonable logical calculus should have had such a kind of inductive definition based on the application of correct rules. In his seminal paper [Gir87] Girard changed this point of view introducing *proof nets*.

The definition of a proof net is no more inductive, but it splits in two distinct sequential phases: (i) Starting from axiom links, by free application of a set of logical rules (logical links), we construct a graph (more precisely a hypergraph) called *proof structure* whose correctness is not guaranteed. (ii) By a suitable *correctness criterion*, we test whether the previously built proof structure is correct or not. Namely, if it is a *proof net*. Girard proposed an exponential algorithm to check correctness of proof structures, successively simplified in the well-known *Danos-Regnier criterion* [DR89] based on a topological approach. Successively, Lafont [Laf95] attacked the problem of correctness of pure multiplicative proof structures in a complete different perspective. Lafont's idea was to give a parsing algorithm to check correctness, that is, a rewriting system of proof structures enriched by a new kind of link (called parsing box). Lafont's solution works for pure multiplicative constant and weakening free nets only. The main reason of this fact is that Lafont deals with nets without an "a priori" weakening and $\perp$ box assignment. Lafont observed that, due to the presence of disconnected components caused by $\perp$ and weakening links (in the following, we will frequently use just weakening links to refer to both), *"there is no hope to find a good parsing algorithm for the full multiplicative fragment"* [Laf95, p. 239].

To overcome such a problem we propose to change the notion of net. Our idea is to have a primitive notion of exponential box but we eliminate the necessity of weakening boxes. This is possible as, for any given MELL sequent proof, it is (semantically) sound to permute its weakenings towards its axioms, so we directly connect weakening formulas to proof net axioms. As a result, our proof nets are always connected. Consequently, we are able to give a complete set of rewriting rules to parse full multiplicative and exponential (MELL) proof nets. We claim that our formulation of proof nets is a good alternative of the classical one not simply a technical escape from the problem. Such an approach might also be seen as a specialization of the *probe* technique of Banach [Ban95]. Anyhow, differently from Banach, we do not need any new extra-logical link.

The structure of the paper: Section 2 defines $\mathsf{MELL}^{\text{-w}}$, a weakening free formulation of MELL. Section 3 introduces the $\mathsf{MELL}^{\text{-w}}$ proof nets. Section 4 states the Danos-Regnier criterion. Section 5 defines the parsing rewriting system. Section 6 proves the equivalence between the parsing system and the Danos-Regnier criterion. Section 7 shows the adequacy of the $\mathsf{MELL}^{\text{-w}}$ proof nets.

2 Permutations: the calculus $\mathsf{MELL}^{\text{-w}}$

The classical sequent calculus for the multiplicative $(\otimes, \parr, \perp, 1)$ and exponential $(?, !)$ fragment of linear logic (MELL) has two kinds of weakening rules:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, W? \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\, W\perp$$

The $W?$ rule permutes with any other rule according to the following scheme:

$$\cfrac{\cfrac{\vdash \Gamma \quad (\vdash \Delta)}{\vdash \Sigma}\, *}{\vdash \Sigma, ?A}\, W? \qquad \text{permutes to} \qquad \cfrac{\cfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\, W? \qquad (\vdash \Delta)}{\vdash \Sigma, ?A}\, *$$

also in the case in which $*$ is an of-course introduction rule. However, the previous permutations do not hold taking $W\bot$ in place of $W?$. In fact:

$$\dfrac{\dfrac{\vdash ?\Gamma, B}{\vdash ?\Gamma, !B}\,!}{\vdash ?\Gamma, !B, \bot}\,W\bot \qquad \text{permutes to} \qquad \dfrac{\dfrac{\vdash ?\Gamma, B}{\vdash ?\Gamma, B, \bot}\,W\bot}{\vdash ?\Gamma, !B, \bot}\,!$$

would introduce a rule (the last one) which violates the side condition of the ! rule. On the other hand, even if the previous instance of an ! rule is syntactically wrong, it is semantically sound. More generally, the rule:

$$\dfrac{\vdash \Gamma, B}{\vdash \Gamma, !B}\,!_\bot$$

where each formula in Γ is a why-not formula or a $\bot$, is semantically correct, since it is derivable in MELL. Replacing the ! rule of MELL with the $!_\bot$ rule, both $W?$ and $W\bot$ can be pushed towards the axioms and eventually merged with them. We obtain in this way a variant of MELL that we call $\mathsf{MELL}^{\text{-w}}$, which is like MELL except for:

(i) In $\mathsf{MELL}^{\text{-w}}$ the introduction rule for ! is $!_\bot$.
(ii) The rules $W?$ and $W\bot$ of MELL are dropped.
(iii) The axioms of $\mathsf{MELL}^{\text{-w}}$ are (Γ is a sequence of $\bot$ or why-not formulas):

$$\vdash p, p^\bot, \Gamma \qquad\qquad \vdash \mathbf{1}, \Gamma$$

The key point of $\mathsf{MELL}^{\text{-w}}$ is that it is a *weakening free* calculus.

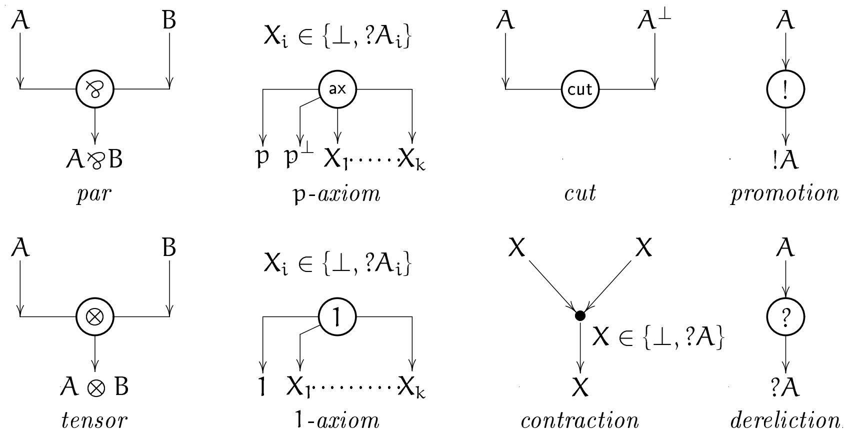
3 Proof Structures

According to $\mathsf{MELL}^{\text{-w}}$, we reshape *proof structures* and *proof nets*. As usual, (at least in the last years) we represent them as hypergraphs (see [Gue96,Reg92]). Their differences w.r.t the classical ones (i.e., as defined by Girard) are:

(i) The $\bot$ formulas may be auxiliary doors of the exponential boxes (as a consequence we shall have an explicit link to contract $\bot$ formulas).
(ii) There are not weakening boxes.
(iii) The axiom links have variable arity.

Remark 1 *The use of a contraction rule for $\bot$ formulas could be avoided at the level of presentation of these notes. It turned to be mandatory if we would study the dynamics of the* $\mathsf{MELL}^{\text{-w}}$ *proof nets or to prove cut elimination.*

A $\mathsf{MELL}^{\text{-w}}$ *link* is a hyperarc labeled by a *type*, one of the MELL connectives or constants $\{\mathsf{ax}, \mathsf{cut}, \otimes, \parr, !, ?, \bullet, \perp, 1\}$ (we use $\bullet$ to denote the *contraction*).



p is an atomic formula — $\mathsf{A}, \mathsf{A}_i, \mathsf{B}$ are arbitrary MELL formulas

Fig. 1. $\mathsf{MELL}^{\text{-w}}$ links.

A $\mathsf{MELL}^{\text{-w}}$ *structure* G is a directed hypergraph whose hyperarcs are $\mathsf{MELL}^{\text{-w}}$ links, and whose vertices are occurrences of formulas. The tail of a link of G is the ordered set of its premises; its head is the ordered set of its conclusions. The number and the shape of the premises/conclusions of a link are constrained by its type, see Figure 1. Each formula of G is conclusion of exactly one link and premise of at most one link; no formula of G may be at the same time premise and conclusion of the same link (this restriction is relevant only for the $\bullet$ links). For any formula A of G, the link *above* A is the link whose conclusion is A; the link *below* A is the link a premise of which is A. The formulas Γ which are not premise of any link of G are the conclusions of G, written $\mathsf{G} \vdash \Gamma$.

Remark 2 *It is crucial for the proposed approach the elimination of explicit links (without premises) introducing weakening formulas. All the weakening formulas are instead introduced by axiom links. In such a way we ensure connectedness of our* $\mathsf{MELL}^{\text{-w}}$ *proof nets and we shall apply the Danos-Regnier correctness criterion without the need to refer to connected components.*

A sub-structure H of a structure G is determined by the set of its links. So, the usual set operations will be used to compose and compare structures. In addition, by G^{x} we shall denote the set of the links of type x contained in G.

3.2 Boxes

A *box* B is a structure in which all the conclusions are why-not or bottom formulas but one, its *principal door*, which is an of-course formula; the why-not or $\perp$ conclusions of B are its *auxiliary doors*. No auxiliary door of a box can be the conclusion of a $\bullet$ link (see Figure 2). The ! link $\mathfrak{l}$ whose conclusion is the principal door of B is its *principal door link*, that is, $\text{PDL}(B) = \mathfrak{l}$.

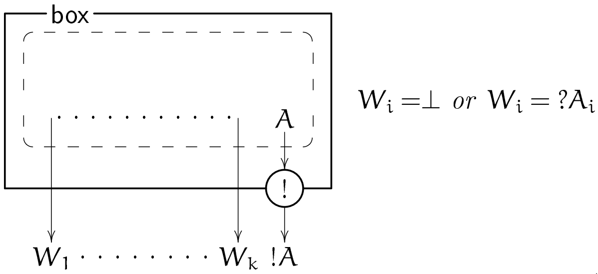


Fig. 2. Box.

Remark 3 *Allowing auxiliary doors of boxes to be $\perp$ formulas is fully justified by the $!_\perp$ rule of* $\mathsf{MELL}^{\text{-w}}$.

3.3 Proof structures

A $\mathsf{MELL}^{\text{-w}}$ proof structure $\mathscr{G}$ with conclusions Γ (written $\mathscr{G} \vdash \Gamma$) is a pair formed by a $\mathsf{MELL}^{\text{-w}}$ structure $G \vdash \Gamma$, and of a boxing map, assigning to each $\mathfrak{l} \in G^!$ a box $B_\mathfrak{l}$, with $\text{PDL}(B_\mathfrak{l}) = \mathfrak{l}$. Boxes have to satisfy the so-called *box nesting condition*, that is, two distinct boxes may nest but not partially overlap. More formally, the set $\text{BOX}(G) = \{B_\mathfrak{l} \mid \mathfrak{l} \in G^!\}$ of the boxes of G satisfies the box nesting condition when: for any pair $B_1, B_2 \in \text{BOX}(G)$, if neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$, then $B_1 \cap B_2 = \emptyset$. Anyhow, according to such a definition, distinct boxes may share one or more auxiliary doors.

The inclusion relation among structures naturally extends to proof structures. Namely, $\mathscr{H} \subseteq \mathscr{G}$ if $H \subseteq G$ and $\text{BOX}(H) \subseteq \text{BOX}(G)$. The box nesting condition also ensures that to any box $B \in \text{BOX}(G)$ corresponds a proof sub-structure $\mathscr{B}$, the *proof box* of B, defined taking $\text{BOX}(B) = \{B' \in \text{BOX}(G) | \text{PDL}(B') \in B\}$.

4 Danos-Regnier correctness criterion

To build the switch structures by which we shall characterize proof nets, we add three new kinds of links: (*i*) the net link, and (*ii*) the switched $\invamp$ and

(iii) the switched $\bullet$ links. The net link has no premise and an arbitrary non-empty set of conclusions. The switched links are instead obtained from a corresponding $⅋$ or $\bullet$ link marking as erased all its premises but one (so, they have only one premise and one conclusion).

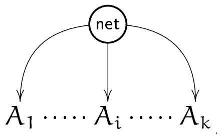


Fig. 3. Net link.

A switching pair for a $\mathsf{MELL}^{\text{-w}}$ proof structure $\mathscr{G}$ is a pair (S_0, S_1) in which $S_0 \subseteq \text{BOX}(G)$ and S_1 is a set of formulas obtained choosing a premise for each $⅋$ and any $\perp$ link of G. Let (S_0, S_1) be a switching pair for $\mathscr{G}$, the corresponding *switch* of $\mathscr{G}$ is the structure S obtained from G replacing each proof box $\mathscr{B} \vdash \Gamma$ corresponding to $B \in S_0$ by a net link with conclusions Γ, and by replacing each $l \in G^{⅋} \cup G^{\bullet}$ by the corresponding switched link obtained marking as erased the premises of l not in S_1. Note that the conclusions Δ of S are the conclusions of G plus the premises of $⅋$ and $\bullet$ links not in S_1.

To any switch $S \vdash \Delta$ we associate an undirected graph S^u with a root node for any conclusion of S by: (i) replacing each link of S by a node; (ii) replacing each formula A of S by an edge connecting the link above A to a root of S^u, when $A \in \Delta$, or by an edge connecting the links above and below A, otherwise. A switch S is acyclic if S^u is; it is connected if S^u is.

Definition 4 (DR-correct structures) *A* $\mathsf{MELL}^{\text{-w}}$ *proof structure* $\mathscr{G}$ *is* DR-correct *if each switch of* $\mathscr{G}$ *is acyclic and connected.*

Definition 5 ($\mathsf{MELL}^{\text{-w}}$ proof net) *A* $\mathsf{MELL}^{\text{-w}}$ *proof structure is a* $\mathsf{MELL}^{\text{-w}}$ proof net *if it is* DR-*correct.*

5 Parsing

The DR-correctness is a topological characterization of $\mathsf{MELL}^{\text{-w}}$ proof nets. We know (and we shall prove) that any $\mathsf{MELL}^{\text{-w}}$ proof net is the image of (at least) a $\mathsf{MELL}^{\text{-w}}$ derivation (modulo some permutations of rules). Namely, that any $\mathsf{MELL}^{\text{-w}}$ proof net may be *sequentialized.* We shall show that the inductive definition corresponding to such a sequentialization induces a parsing (graph) grammar σ for $\mathsf{MELL}^{\text{-w}}$ proof structures accepting $\mathsf{MELL}^{\text{-w}}$ proof nets only.

A *parsing* $\mathsf{MELL}^{\text{-w}}$ proof structure is a $\mathsf{MELL}^{\text{-w}}$ proof structure whose hypergraph may also contain net links (but not switched links), that is, they are the intermediate structures obtained applying the σ-grammar. The definitions of switch and DR-correctness naturally extend to parsing proof structure.

Definition 6 (σ-grammar) *The* σ-grammar *is the graph grammar given by the rewriting rules of Figure 4, with the proviso that an instance* r *of the l.h.s. of a rule is a* σ*-redex (and then it can be contracted) only if the following two side-conditions hold:*

(i) No border of a box splits r *in two non-empty parts, that is, for any box* B*, if* $r \cap B \neq \emptyset$*, then* $r \subseteq B$*.*
(ii) If r *is a* $\otimes$ *or* cut *redex (i.e., a redex for the rule scanning a* $\otimes$ *or* cut *link), then the two* net *links in* r *are distinct.*

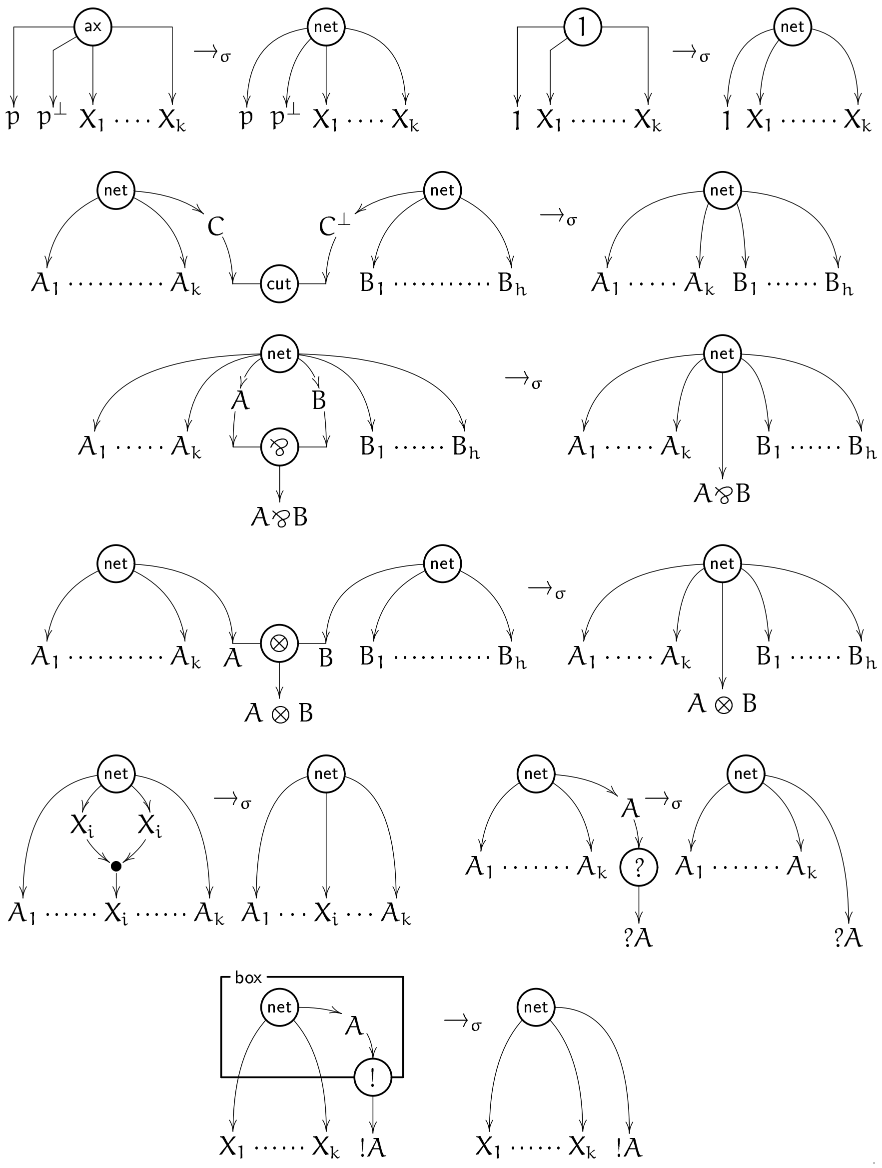


Fig. 4. The rules of the σ-grammar.

Each rule of σ contracts a redex to a net link (since we do not consider cut elimination, there is no ambiguity in saying redex or reduction dropping the prefix σ). In other words, net links play the role of non-terminal symbols and each rule of σ corresponds to the scanning of a $\mathsf{MELL}^{\text{-w}}$ link. Hence, since a redex is uniquely determined by the $\mathsf{MELL}^{\text{-w}}$ link it scans, a reduction $\rho = l_0 l_1 \ldots l_k$ will be denoted by the sequence of the links it scans.

The syntax category corresponding to net links composes of the $\mathsf{MELL}^{\text{-w}}$ proof structures $\mathscr{G} \vdash \Gamma$ which reduce to a parsing structure formed of a net link with conclusions Γ (let us denote such a structure by net_Γ).

Definition 7 (σ-correctness) *A (parsing)* $\mathsf{MELL}^{\text{-w}}$ *proof structure* $\mathscr{G} \vdash \Gamma$ *is* σ-correct *if* $\mathscr{G} \to_\sigma^* \mathsf{net}_\Gamma$.

6 Equivalence of the correctness criteria

Lemma 8 *Let* $\mathscr{P} \vdash \Gamma$ *be a parsing* $\mathsf{MELL}^{\text{-w}}$ *proof structure with no* $!$ *link. If* $\mathscr{P}$ *is* DR*-correct and is not* net_Γ*, then it contains at least a* σ*-redex.*

PROOF. Let us assume that $\mathscr{P}$ does not contain a redex scanning an axiom, a $⅋$, a $\bullet$, or a $?$ link. Namely, that $\mathscr{P}$ might only contain redexes for $\otimes$ or cut links. Our aim is to prove that, if $\mathscr{P}$ is DR-correct and is not net_Γ, then it contains at least a $\otimes$ or a cut link whose premises are conclusions of two distinct net links. We see that $\mathscr{P}$ does not contain any axiom link, and that no $?$ link of $\mathscr{P}$ is below a conclusion of a net link. We claim that any net link of $\mathscr{P}$ has at least a conclusion which is the premise of a $\otimes$ or a cut link (because of the DR-correctness). Hence, let S be a switch of $\mathscr{P}$. Let us consider the set X of the $\otimes$ and cut links a premise of which is conclusion of a net link of S. Since $\mathscr{P}$ is DR-correct, there is no link $l \in \mathsf{X}$ whose premises are both conclusions of the same net link. Then, to prove that $\mathscr{P}$ contains a redex it suffices to show that there exists $l \in \mathsf{X}$ whose premises are both conclusions of net links. Let us proceed by *reductio ad absurdum*, showing that if such an $l \in \mathsf{X}$ does not exist, then S contains a cycle. By the previous claim, for any net link n there is a conclusion A s.t. the link l below it is in X. If B is the other premise of l, let ϕ be the maximal ascending path of S starting from B (a sequence $A_0 l_1 A_1 \ldots A_{i-1} l_i A_i \ldots$ of formulas A_i and links l_i is an ascending path when A_{i-1} is the conclusion of l_i and A_i one of its premises). By hypothesis ϕ is not empty and the path $\psi = AlB\phi$ of S^u connects a conclusion of n to the conclusion of another net link. The last link of ψ is not in X, since ϕ is maximal and we are assuming that there is no link of X whose premises are both conclusions of a net link. So, starting from a net link n_0, we find a path ψ_0 connecting the conclusions A_0 of n_0 to a conclusion C_0

of a net link n_1; proceeding from n_1, we find a path ψ_1 that concatenated to ψ_0 gives the path $\psi_0 n_1 \psi_1$ connecting A_0 to the conclusion C_1 of a net link n_2 (the path is correct since the last link of ψ_0 and the first link of ψ_1 are definitely distinct); and so on building a sequence of net links $n_0, n_1, \ldots, n_i$ crossed by the path $\psi_0 n_1 \ldots n_{i-1} \psi_i$ connecting n_0 to n_1. But, since S is finite, we eventually find an $i > 0$ for which $n_j = n_i$, with $j < i$, that is, we get a path of S^u which is a cycle, contradicting the DR-correctness of $\mathscr{P}$. □

Theorem 9 (equivalence) *A (parsing)* $\mathsf{MELL}^{\text{-}\mathsf{w}}$ *proof structure* $\mathscr{P}$ *is* DR-*correct iff it is* σ-*correct.*

PROOF. By inspection of the rules of σ, we see that the DR-correctness is invariant under σ-reduction. So, if $\mathscr{P} \to_\sigma^* \mathsf{net}$, then $\mathscr{P}$ is DR-correct. Let us prove the converse proceeding by induction on the number of boxes of $\mathscr{P}$. The base case is proved by Lemma 8. For the induction case, let us take the proof sub-structure $\mathscr{B}'$ obtained by a proof box $\mathscr{B}$ removing its principal door link. By repeated application of the induction hypothesis, we see that $\mathscr{P} \to_\sigma^* \mathscr{R}' \to_\sigma \mathscr{R} \to_\sigma^* \mathsf{net}$ where $\mathscr{R}'$ and $\mathscr{R}$ are the parsing proof structures obtained from $\mathscr{P}$ putting a net link in place of $\mathscr{B}'$ and $\mathscr{B}$, respectively. □

Corollary 10 (unique normal form) *The* σ-*grammar is strongly normalizing and* net_Γ *is the unique normal form of any* $\mathsf{MELL}^{\text{-}\mathsf{w}}$ *proof net* $\mathscr{G} \vdash \Gamma$.

PROOF. Let $\mathscr{G} \to_\sigma \mathscr{P}$. We have that: (i) The size of $\mathscr{P}$ is smaller than the one of $\mathscr{G}$; (ii) $\mathscr{P}$ is DR-correct; (iii) $\mathscr{P} \vdash \Gamma$. So, there is no infinite reduction of $\mathscr{G}$ and, by Theorem 9, net_Γ is the unique normal form of $\mathscr{G} \vdash \Gamma$. □

7 Adequacy and sequentialization

So far we have got a new correctness criterion for proof structures that we have proved equivalent to the topological one of Danos-Regnier . On the other hand our proof nets are not standard. So, we have to prove that they are adequate for $\mathsf{MELL}^{\text{-}\mathsf{w}}$. Namely, that for any $\mathsf{MELL}^{\text{-}\mathsf{w}}$ proof Π with conclusions $\vdash \Gamma$ (let us denote it $\Pi \vdash \Gamma$) there is a corresponding $\mathsf{MELL}^{\text{-}\mathsf{w}}$ proof net with the same conclusions.

Theorem 11 (adequacy) *Let* $\Pi \vdash \Gamma$ *be a* $\mathsf{MELL}^{\text{-}\mathsf{w}}$ *proof. There is a* $\mathsf{MELL}^{\text{-}\mathsf{w}}$ *proof net* $\mathscr{G} \vdash \Gamma$ *with a link for each inference rule of* Π.

PROOF. By an easy induction on the construction of Π. Some care is required just in the treatment of the ! rule, because of the restriction that no auxiliary door of a box can be conclusion of a contraction link. $\square$

Theorem 12 (sequentialization) *For any* $\mathsf{MELL}^{\text{-w}}$ *proof net* $\mathscr{G} \vdash \Gamma$ *there is a* $\mathsf{MELL}^{\text{-w}}$ *proof* $\Pi[\pi] \vdash \Gamma$ *effectively constructed via* $\pi : \mathscr{G} \to_{\sigma}^{*} \mathsf{net}_{\Gamma}$.

PROOF. Let us start observing that, if $\pi : \mathscr{G} \to_{\sigma}^{*} \mathscr{P}$, then for any $l \in \mathsf{P}^{\mathsf{net}}$ there is: (i) a proof net $\mathscr{N}_l \subseteq \mathscr{G}$ with conclusions Γ_l s.t. $\mathscr{G}$ is obtained replacing each $l \in \mathsf{P}^{\mathsf{net}}$ with the corresponding $\mathscr{N}_l$; (ii) a reduction $\pi_l \subseteq \pi$ (i.e., π is obtained by π_l erasing some of its redexes) s.t. $\pi_l : \mathscr{N}_l \to_{\sigma}^{*} \mathsf{net}_{\Gamma_l}$. The proofs of such observations are by induction on the length of $\pi = l_1 \ldots l_k$ and by case analysis on the type of l_k. Hence, let us proceed by induction on the size of $\mathscr{G}$. The base case is direct: $\mathscr{G}$ composes of an axiom link only. For the other cases, let $\pi : \mathscr{G} \to_{\sigma}^{*} \mathscr{R} \to \mathsf{net}_{\Gamma}$. By the initial observations, we can associate to each net link $l \in \mathsf{R}^{\mathsf{net}}$ a σ-correct proof sub-structure of $\mathscr{G}$ and then, by the induction hypothesis, a $\mathsf{MELL}^{\text{-w}}$ proof with the same conclusions. Hence, replacing the net links in the redex R by their corresponding $\mathsf{MELL}^{\text{-w}}$ proof, and replacing the MELL link of R by an inference rule of the same type, we get the $\mathsf{MELL}^{\text{-w}}$ proof $\Pi[\pi]$ we are looking for. The way in which $\Pi[\pi]$ is built shows that it contains an inference rule for each link of $\mathscr{G}$ and that the order in which such rules are applied accords to the order in which the corresponding links are scanned by π. $\square$

8 Conclusions

There is a natural two-way mapping between $\mathsf{MELL}^{\text{-w}}$ and MELL proof structures according to the permutations described in Section 2 (because of such permutations the previous mapping cannot however be a bijection). Given a $\mathsf{MELL}^{\text{-w}}$ proof structure $\mathscr{G}_{-}$ we obtain a Girard proof structure $(\mathscr{G}_{-})^{+}$ by: (i) choosing a weakening formula X which is conclusion of an axiom link a; (ii) replacing the connection of X to a with a box containing all the boxes having X as an auxiliary door; (iii) iterating the steps $(i$-$ii)$ until there are no more X's. Vice versa, given a Girard proof structure $\mathscr{G}$ we obtain a $\mathsf{MELL}^{\text{-w}}$ proof structure $(\mathscr{G})^{-}$ just replacing the link l above each weakening formula X of $\mathscr{G}$ with a direct connection between X and an axiom a contained in the box of l. Correctness is invariant under the previous translation from $\mathsf{MELL}^{\text{-w}}$ to MELL proof structures, but not under the mapping going in the opposite direction. In fact, given a MELL proof net $\mathscr{G}$, each proof structure $\mathscr{G}^{-}$ is definitely correct, but $\mathscr{G}^{-}$ may be correct also in the case that $\mathscr{G}$ is a proof structure with a wrong assignment of weakening boxes—even if all the $\mathscr{G}^{-}$ are correct we could not state that $\mathscr{G}$ is a proof net. To solve such a problem we could reformulate the σ grammar for MELL giving for weakening boxes a rule similar to the one proposed for the exponential boxes. Nevertheless, since we think that weakening boxes are unnatural, we do not like such a solution and in our approach we replace weakening boxes with the minimal information required to get a correct sequentialization, if any. Lafont too implicitly shows in his paper [Laf95] his dislike with respect to weakening boxes neglecting them at all. The main consequence of this disregard is the increase of the cost of the validation of nets. In fact, since in the constant only multiplicative fragment (no atomic symbols but the constants) the provability problem can be reduced to the proof structure correctness problem, and since such a fragment is NP-complete as the multiplicative one [Lin95], if we do not use weakening boxes at all there is no hope to get a polynomial parsing algorithm in the presence of constants. The latter is the first main reason because of which we claim that our solution is not only a technical escape. In fact, the cost of the validation of a proof net cannot be comparable with the cost of the search of a proof ending with its conclusions. So, we propose the σ grammar giving a quadratic algorithm to validate proof nets: any accepting reduction of a proof net with n links has length n, but at each step a search linear in the current size of the structure is required to get the next redex to be reduced. The second reason because of which we support our choice is connected with the implementation of cut elimination. In fact, the use of exponential boxes can be avoided indexing each formula by a level (see [MM95]) which may be interpreted as the box nesting depth of the formula [Gue96,GMM96a]. A parsing grammar can then be given also for such leveled proof nets without boxes. Such a grammar, suitably extended to implement a mark and sweep algorithm for garbage collection, is the key point used for the local and distributed implementation of the cut elimination we studied with Martini [GMM96b].

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