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Lifting Complete Orders to Achieve co-Additivity of Closure Operators

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Abstract

We define the notion of meet-uniformity for closure operators on a complete lattice, which corresponds to co-additivity restricted to subsets mapped into the same element, and we study its properties. A class of closures given by principal filters and the downward closures are relevant examples of meet-uniform closures. Next, we introduce a lifting of a complete order by means of a meet-uniform closure. Our main results show that this lifting preserves the complete lattice structure, and allows the meet-uniform closure to become fully co-additive.

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1 Introduction

We introduce the notion of *meet-uniform* (upper) closure operator on a complete lattice, and prove that meet-uniformity is sufficient to achieve co-additivity of these closure operators, relatively to an induced complete order. A meet-uniform closure on a complete lattice C is a closure operator (viz. a monotone, idempotent and extensive operator) such that if $Y \subseteq C$ is a subset of elements all mapped to some $\bar{z}$, then also its infimum $\wedge_C Y$ is mapped to $\bar{z}$.

It is well known that closure operators are, in general, neither additive nor co-additive, and this holds even if, in addition, they are meet-uniform. However, meet-uniformity provides a systematic way for lifting the complete order, maintaining a complete lattice structure, achieving co-additivity. In fact, we prove that meet-uniform closures can always be lifted to co-additive closures, by refining the complete order. We study the lattice of all meet-uniform closure operators on a complete lattice, and provide some remarkable examples of meet-uniform (non co-additive) closure operators. The results of the paper can be naturally extended to *lower* closure operators by duality.

This paper is originated by studying closure operators in theoretical computer science, where closure operators are extensively used, notably in semantics of programming languages (e.g., see [4, 9]). The achieved co-additivity of meet-uniform closures may provide an alternative "order-theoretic" interpretation of the relation between properties of semantics at different levels of abstraction. Indeed, semantics for programming languages can be specified by suitable closure operators. Cousot and Cousot proved in [4] that most semantic definitions for programming languages can be systematically derived by closure operators on the lattice of execution traces of a transition system specifying the operational semantics of a program. These include, for instance, the more abstract *erratic*, *angelic* and *demoniac* relational semantics of transition systems, and the *denotational* semantics. More abstract semantic denotations are here obtained by *abstract interpretation* ([3]) as the closed elements (fixpoints) of a corresponding closure operator on sets of execution traces. In this framework, the *identical abstraction*, leaving unchanged the domain of the computation, corresponds to the identity operator which is the bottom closure operator, while the *straightforward abstraction* which forgets about program computations, corresponds here to the constant function mapping any set of traces to the top element, which is precisely the top closure operator. In this formulation, the lattice of semantic definitions is therefore isomorphic to the lattice of closure operators on the lattice of execution traces. However, there are some specific properties of each semantic definition which cannot be compared according to their level of abstraction (e.g., see the property of being a model for an inductive definition in Example 1.1 below). We claim that meet-uniform closures play here a central rôle in order to fix a lattice-theoretic setting, which is based on the lifted complete order, where properties of semantic definitions at different levels of abstraction can be compared to each other. This provides a generalization of the *model-theoretic reconstruction of the operational semantics of logic programs* introduced in [7], to arbitrary semantic definitions for programming languages. The following example shows this fact for positive inductive definitions, and gives the intuition on the lifted complete order for meet-uniform closures. It is worth noting that inductive definitions form the standard framework for operational semantic specification of programming languages, and most semantic definitions in [4] are obtained by abstracting inductive definitions.

Example 1.1 Consider an arbitrary positive inductive definition $\Im$, i.e., a pair $\langle U, \Phi\rangle$ where U is any set (the universe) and Φ is a set of rule instances, i.e., pairs $P \to c$ such that $P, \{c\} \subseteq U$ (cf. [1]). It is clear that the mapping $sat_{\Im}$ which extends any $S \subseteq U$ into the least set $W \subseteq U$ such that $S \subseteq W$, and $\forall P \to c \in \Phi$, $(P \subseteq W) \Rightarrow (c \in W)$, is a closure operator on the subsets of the universe ordered by inclusion. $sat_{\Im}(S)$ is called a $\Im$-*model* (or $\Im$-*closed set*) for the inductive definition $\Im$. We adopt the notation in [4] and denote $\bar{\Phi}(S)$ the set $\{c \in U \mid \exists P \subseteq S.\ P \to c \in \Phi\}$, for $S \subseteq U$. As observed in [4], the least fixpoint of $\bar{\Phi}$, if it exists, is the least $\Im$-model, and ensures that $\Im$ is well defined. Also, it is immediate that $sat_{\Im}(S) = S$ iff $\bar{\Phi}(S) \subseteq S$.
Let $\rho : \wp(U) \to \wp(U)$ be a closure operator (if $\rho(S) = S$ then S is called an *abstract object*), and $\Im = \langle U, \Phi\rangle$ and $\Im' = \langle U, \Phi'\rangle$ be inductive definitions such that for all $S \subseteq U$, $\bar{\Phi}'(\rho(S)) = \rho(\bar{\Phi}(S))$ and $(sat_{\Im'}(S) = S) \Rightarrow (\rho(S) = S)$ (i.e., every $\Im'$-model is an abstract object). In this case, we say that $\Im'$ is *more abstract* than $\Im$. It is easy to verify that for any $S \subseteq U$, $(sat_{\Im}(S) = S) \Rightarrow (sat_{\Im'}(\rho(S)) = \rho(S))$. Meet-uniformity is a crucial property here in order to extend the set of $\Im$-models to include $\Im'$-models, and to compare both kinds of models with respect to the corresponding expressive power. Indeed, by the above assumptions and observations, the set $\mathcal{J} = \{S \subseteq U \mid sat_{\Im'}(\rho(S)) = \rho(S)\}$ contains both $\Im$-models and $\Im'$-models: S is in $\mathcal{J}$ iff when abstracted (by ρ) it is a $\Im'$-model. However, $\mathcal{J}$ is not, in general, a complete lattice (w.r.t. $\subseteq$), in particular $\mathcal{J}$ may not have a least (canonical) element. On the other hand, if ρ is co-additive then $\mathcal{J}$ is a complete lattice, and thus both kinds of models can be uniformly compared, and a canonical representative model can be chosen. We claim that, whenever ρ is meet-uniform, it is always possible (by Theorem 5.12 below) to lift the order (viz. inclusion) so as $\mathcal{J}$ becomes a complete lattice. The new order among models in $\mathcal{J}$ reflects precisely the existing relation between inductive definitions (namely that $\Im'$ is more abstract than $\Im$): notably, the least $\Im'$-model is also the least (viz. most abstract) element in $\mathcal{J}$. In this case, $\mathcal{J}$ inherits on models the existing lattice structure of semantics. The key point here is that the new order has to reflect adequately the property of a ρ-closed (abstract) element $S \subseteq U$ being representative of any S' such that $\rho(S') = S$. Meet-uniform closure operators allow this order-extension to be consistently applied. $\square$

A similar approach to the example above has been firstly applied in [7], where logic and operational models for logic programs (viz., Horn-clause theories) are compared by a partial order. Some of the results in [7] follow immediately in our framework by meet-uniformity of the involved closure operators. This because the operational models of *computed answers* and *correct answers* of a logic program can be both specified by inductive definitions, which are related by a meet-uniform closure operator, as above. This operation is an instance of a *downward closure*, later discussed in Section 4.2.

The plan of the paper is as follows. In Section 2, some basic notions for closure operators are recalled. In Section 3, we introduce the notion of meet-uniformity, and, for closure operators, we study its properties. In Section 4, we consider examples of meet-uniform closures: a class of principal filters, and downward closures are shown to satisfy the property of meet-uniformity. In Section 5, we define the lifting of a complete order via a meet-uniform closure operator, and we show that this process preserves the complete lattice structure and allows the meet-uniform closure to get co-additivity.

2 Preliminaries

In this section, we briefly introduce some notation used throughout the paper and summarize some definitions and well-known properties concerning closure operators (for more details see [2, 11, 12]).

Let C and D be sets. The powerset of C is denoted by $\wp(C)$, and its cardinality by $|C|$. The set-difference between C and D is denoted by $C \setminus D$. If f is a function defined on C and $D \subseteq C$ then $f(D) = \{f(x) \mid x \in D\}$. Functions will be sometimes denoted by Church's lambda notation. The set C equipped with a partial order $\leq$ is denoted by $C_{\leq}$. We use $x < y$ to denote that $x \leq y$ and $x \neq y$. A complete lattice C with partial ordering $\leq$, least upper bound (*lub*) $\vee$, greatest lower bound (*glb*) $\wedge$, top element $\top = \wedge\emptyset = \vee C$, and bottom element $\bot = \vee\emptyset = \wedge C$, is denoted by $\langle C, \leq, \vee, \wedge, \top, \bot\rangle$. We will often abuse notation by denoting lattices with their poset notation. Sometimes, we will use subscripts to disambiguate lattice operations (e.g., $\leq_C$ and $\wedge_C$). A function $f : C \to D$ between complete lattices is *additive* if for any $Y \subseteq C$, $f(\vee_C Y) = \vee_D f(Y)$. *Co-additivity* is dually defined.

An *(upper) closure operator* on the poset C is an operator $\rho : C \to C$ monotonic, idempotent and extensive (viz. $\forall x \in C.\ x \leq \rho(x)$). We denote by *uco*$(C)$ the set of all closure operators on the poset C. If C is a complete lattice then each closure operator ρ is uniquely determined by the set of its fixpoints, which is its image $\rho(C)$. A subset $Y \subseteq C$ is the set of fixpoints of a closure operator iff Y is a *Moore-family* of C, i.e. $\top \in Y$ and for any $X \subseteq Y$, with $X \neq \emptyset$, $\wedge X \in Y$. Furthermore, the set of fixpoints $\rho(C)$ is a complete lattice with respect to the order of C, but, in general, it is not a complete sublattice of C, since the *lub* in $\rho(C)$ might be different from that in C. If ρ is additive then $\rho(C)$ is a complete sublattice of C. In the following, we will often denote a closure operator by the set of its fixpoints. We denote by $\langle uco(C), \sqsubseteq, \sqcup, \sqcap, \lambda x.\top, \lambda x.x\rangle$ the complete lattice of all closure operators on the complete lattice C, where for every $\rho, \eta \in uco(C)$, $\{\rho_i\}_{i\in I} \subseteq uco(C)$ and $x \in L$:

- $\rho \sqsubseteq \eta$ iff $\forall x \in C.\ \rho(x) \leq \eta(x)$, or equivalently, $\rho \sqsubseteq \eta$ iff $\eta(C) \subseteq \rho(L)$;
- $(\sqcup_{i\in I}\rho_i)(x) = x \Leftrightarrow \forall i \in I.\ \rho_i(x) = x$;
- $(\sqcap_{i\in I}\rho_i)(x) = \wedge_{i\in I}\rho_i(x)$;
- $\lambda x.\top$ (viz. the operator mapping all the elements to $\top$) is the top element in *uco*(C), whereas $\lambda x.x$ (viz. the identity operator on C) is the bottom element.

For a closure operator $\rho \in uco(C)$ and $Y \subseteq C$, the following two properties hold:

(i) $\rho(\wedge_{y\in Y}\rho(y)) = \wedge_{y\in Y}\rho(y)$;

(ii) $\rho(\vee Y) = \rho(\vee_{y\in Y}\rho(y))$.

It is also known that *uco*(C) is *dual atomistic*, namely each closure different from the top $\lambda x.\top$ is the infimum of the set of dual atoms following it. *Lower closure operators* are dually defined, and for them dual properties to those above hold.

If C and D are posets and $\alpha : C \to D$, $\gamma : D \to C$ are monotonic functions such that $\forall c \in C.\ c \leq_C \gamma(\alpha(c))$ and $\forall d \in D.\ \alpha(\gamma(d)) \leq_D d$, then (γ, D, C, α) is called a *Galois connection* between C and D. We also recall that the above definition of Galois connection is equivalent to that of adjunction: (γ, D, C, α) is an *adjunction* if $\forall c \in C.\forall d \in D.\ \alpha(c) \leq_D d \Leftrightarrow c \leq_C \gamma(d)$.

3 Meet-Uniform Closure Operators

In what follows, we assume C and D be two complete lattices.

Definition 3.1 A function $f : C \to D$ is *meet-uniform* if for all $Y \subseteq C$,

$$(\exists \bar{x} \in Y. \forall y \in Y. f(y) = f(\bar{x})) \Rightarrow (\exists \bar{x} \in Y. f(\wedge_C Y) = f(\bar{x})). \qquad \square$$

In other terms, a mapping f from C into D enjoys the property of meet-uniformity if it is co-additive for any family of elements all mapped by f into the same element. Note that the above condition is vacuously satisfied for $Y = \emptyset$ and Y singleton. Thus, in the proofs of meet-uniformity, we will assume $|Y| > 1$. Further, notice that if f is injective then it is trivially meet-uniform.

The principal aim of this section is to study *meet-uniform closure operators* on complete lattices. Note that the only injective closure is the identity operator. Therefore, injectivity is not relevant to characterize meet-uniformity of closure operators. As the following remark shows, for any closure operator the dual property of *join-uniformity* is always satisfied.

Remark 3.2 *Each $\rho \in uco(C)$ is join-uniform, i.e. for all $Y \subseteq C$, $(\exists \bar{x} \in Y. \forall y \in Y. \rho(y) = \rho(\bar{x})) \Rightarrow (\exists \bar{x} \in Y. \rho(\vee Y) = \rho(\bar{x}))$.*

Proof. $\rho(\vee Y) = \rho(\vee_{y \in Y} \rho(y)) = \rho(\rho(\bar{x})) = \rho(\bar{x})$. ∎

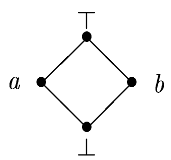
To denote the set of meet-uniform closure operators on C, we use the following notation: $uco^*(C) = \{\rho \in uco(C) \mid \rho \text{ is meet-uniform}\}$. Clearly, the bottom element of $uco(C)$, viz. the identity operator, belongs to $uco^*(C)$. Furthermore, the following result holds.

Proposition 3.3 *$uco^*(C)$ is a Moore-family of $uco(C)$.*

Proof. Clearly, $\lambda x.\top \in uco^*(C)$. Now, let us consider $\{\rho_i\}_{i \in I} \subseteq uco^*(C)$. We show that $\sqcap_{i \in I} \rho_i \in uco^*(C)$. Consider any $Y \subseteq C$ such that $|Y| > 1$, and suppose that there exists $\bar{x} \in Y$ such that for all $y \in Y$, $(\sqcap_{i \in I} \rho_i)(y) = (\sqcap_{i \in I} \rho_i)(\bar{x})$. If we prove that for any $i \in I$, $\rho_i(\wedge Y) = \rho_i(\bar{x})$, the thesis follows, since $(\sqcap_{i \in I} \rho_i)(\wedge Y) = \wedge_{i \in I} \rho_i(\wedge Y) = \wedge_{i \in I} \rho_i(\bar{x}) = (\sqcap_{i \in I} \rho_i)(\bar{x})$. Thus, consider any $j \in I$ and $y \in Y$. Then, $y \leq (\sqcap_{i \in I} \rho_i)(y) = (\sqcap_{i \in I} \rho_i)(\bar{x}) = \wedge_{i \in I} \rho_i(\bar{x})$, and, analougously, $\bar{x} \leq \wedge_{i \in I} \rho_i(y)$. Hence, $y \leq \rho_j(\bar{x})$ and $\bar{x} \leq \rho_j(y)$, from which $\rho_j(y) = \rho_j(\bar{x})$ easily follows. Therefore, for all $i \in I$, $\rho_i(\wedge Y) = \rho_i(\bar{x})$. ∎

Thus, $uco^*(C)$ is the set of fixpoints of a closure operator $\psi^* \in uco(uco(C))$ defined as follows: for any $\rho \in uco(C)$, $\psi^*(\rho) = \sqcap\{\eta \in uco(C) \mid \rho \sqsubseteq \eta, \ \eta \text{ is meet-uniform}\}$. Another consequence of the above proposition is that $uco^*(C)$ is a complete lattice with respect to the order inherited from $uco(C)$, and, in particular, is a complete meet-sublattice of $uco(C)$. Next example shows that, in general, $uco^*(C)$ is not a complete sublattice of $uco(C)$, and therefore, the closure ψ^* on $uco(C)$ is not additive.

Example 3.4 Consider the finite lattice C depicted below.

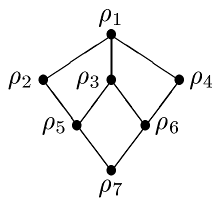


The closure operators on C are the following:

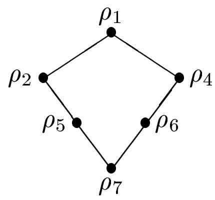
$$\rho_1 = \{\top\}, \ \rho_2 = \{\top, a\}, \ \rho_3 = \{\top, \bot\}, \ \rho_4 = \{\top, b\},$$

$$\rho_5 = \{\top, a, \bot\}, \ \rho_6 = \{\top, b, \bot\}, \ \rho_7 = \{\top, a, b, \bot\}.$$

Thus, $uco(C)$ is the lattice depicted below.



It is immediate to verify that the meet-sublattice $uco^*(C)$ of the meet-uniform closures is the following.



In fact, the only closure which is not meet-uniform is ρ_3: $\rho_3(a) = \rho_3(b) = \top$, whilst $\rho_3(a \wedge b) = \rho_3(\bot) = \bot$. Moreover, observe that $uco^*(C)$ is not a sublattice of $uco(C)$: the join of ρ_5 and ρ_6 in $uco^*(C)$ is clearly different from that in $uco(C)$. $\square$

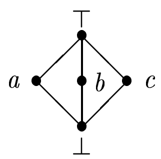
The example above allows one to draw the following additional consequences.

(i) The property of dual-atomicity of $uco(C)$ does not hold anymore for the complete lattice $uco^*(C)$.

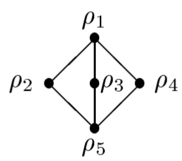
(ii) It is not true that the functional composition of two meet-uniform closure operators (whenever is a closure) is meet-uniform. In fact, consider the meet-uniform closures ρ_5 and ρ_6 of the above example. The composition $\rho_5 \circ \rho_6 = \rho_6 \circ \rho_5$ is defined, and is the closure ρ_3 which is not meet-uniform.

Very recently, it has been shown that for a complete lattice C which is chain inf-distributive (i.e. such that for any chain $Y \subseteq C$ and $x \in C$, $x \wedge (\vee Y) = \vee_{y \in Y}(x \wedge y)$), $uco(C)$ is pseudo-complemented (cf. [8]). Next example shows that in these hypotheses, the complete meet-sublattice $uco^*(C)$ of $uco(C)$ is not, in general, pseudo-complemented.

Example 3.5 Consider the following lattice C.



Consider the following closures on C: $\rho_1 = \{\top\}$, $\rho_2 = \{\top, a, b, \bot\}$, $\rho_3 = \{\top, a, c, \bot\}$, $\rho_4 = \{\top, b, c, \bot\}$, $\rho_5 = C$. It is simple to verify that they are the only meet-uniform closures on C. Hence, $uco^*(C)$ is the lattice below.



Then, clearly, $uco^*(C)$ is not pseudo-complemented (ρ_2, ρ_3 and ρ_4 do not have the pseudo-complement). $\square$

As a side observation, note that meet-uniformity is not preserved by pseudo-complementation: in fact, with reference to the above example, it is possible to verify that the pseudo-complement of ρ_2 is $\rho_2^* = \{\top, c\}$, which is not meet-uniform.

Suppose $f : C \to D$, and consider the corresponding equivalence relation $\approx_f$ induced on C by f, and defined as $x \approx_f y \Leftrightarrow f(x) = f(y)$. The following definition assigns a canonical representative with each equivalence class defined by $\approx_f$.

Definition 3.6 For any $\bar{x} \in C$, define $\nabla(\bar{x}) = \wedge[\bar{x}]_{\approx_f}$ $(= \wedge\{x \in C \mid f(x) = f(\bar{x})\})$. □

The property of meet-uniformity of f allows one to verify that this is a good definition. Indeed, for each $\bar{x} \in C$, $\nabla(\bar{x}) \approx_f \bar{x}$, viz. $f(\nabla(\bar{x})) = f(\bar{x})$. Further, it is worth noticing that if $\rho \in uco(C)$ then $\nabla(\bar{x}) = \nabla(\rho(\bar{x}))$. We will make intensive use of these remarks in the sequel. The following observation provides an alternative characterization of the property of meet-uniformity.

Proposition 3.7 *Assume $f : C \to D$ is monotonic. f is meet-uniform iff for any $x \in C$, $\nabla(x) \approx_f x$.*

Proof. ($\Rightarrow$) Straightforward, by definition of meet-uniformity.
($\Leftarrow$) Consider any $Y \subseteq C$ (with $|Y| > 1$), and assume that there exists $\bar{x} \in Y$ such that $f(y) = f(\bar{x})$, for all $y \in Y$. Obviously, $f(\wedge_C Y) \leq_D f(\bar{x})$. On the other hand, by hypothesis, $\nabla(\bar{x}) \approx_f \bar{x}$, i.e. $f(\bar{x}) = f(\nabla(\bar{x})) \leq_D f(\wedge_C Y)$. Therefore, $f(\wedge_C Y) = f(\bar{x})$. ■

In particular, the above result holds for closure operators: a closure ρ is meet-uniform iff each equivalence class induced by $\approx_\rho$ contains its infimum.

We conclude this section by observing that meet-uniformity can be induced on closure operators by means of Galois connections. By a slight abuse of terminology, we call meet-uniform the following Galois connections.

Definition 3.8 A Galois connection (γ, D, C, α) is *meet-uniform* if α is meet-uniform. □

It is well known that if (γ, D, C, α) is a Galois connection, then $\gamma \circ \alpha$ is a closure operator on C ([6]). Next result shows that meet-uniform Galois connections induce meet-uniform closures.

Proposition 3.9 *The Galois connection (γ, D, C, α) is meet-uniform iff $\gamma \circ \alpha \in uco^*(C)$.*

Proof. We only prove the "only if" part, since the other direction is analogous.
Consider any $Y \subseteq C$ such that $|Y| > 1$, and suppose that there exists $\bar{x} \in Y$ such that $\forall y \in Y.\ \gamma(\alpha(y)) = \gamma(\alpha(\bar{x}))$. Then, for all $y \in Y$, $\alpha(\gamma(\alpha(y))) = \alpha(\gamma(\alpha(\bar{x})))$, and, since $\alpha \circ \gamma \circ \alpha = \alpha$, $\alpha(y) = \alpha(\bar{x})$. Hence, $\alpha(\wedge_C Y) = \alpha(\bar{x})$, from which, $\gamma(\alpha(\wedge_C Y)) = \gamma(\alpha(\bar{x}))$. ■

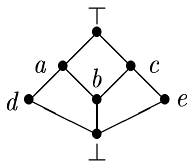
4 Examples of Meet-Uniform Closures

Next two subsections show some interesting families of meet-uniform closure operators.

4.1 Closures by Principal Filters are Meet-Uniform

Let C be a complete lattice. Fix any element $c \in C$, and consider the operator ϕ_c on C defined as follows: for any $x \in C$, $\phi_c(x) = c \vee x$. It is immediate to observe that $\phi_c \in uco(C)$. Clearly, the set of fixpoints of ϕ_c is $\phi_c(C) = \{x \in C \mid c \leq x\}$, i.e. the principal filter generated by c. Obviously, it is not true that for any complete lattice C and $c \in C$, each ϕ_c is a meet-uniform closure. In fact, considering the lattice C of Example 3.5, and the principal filter on C generated by the element $a \in C$, viz. the closure ϕ_a, as said in that example, ϕ_a is not meet-uniform: $\phi_a(b) = \phi_a(c) = \top$, while $\phi_a(b \wedge c) = \phi_a(\bot) = a$. Note that if C is an infinitely sup-distributive lattice, i.e. a complete lattice such that for any $x \in C$ and $Y \subseteq C$, $x \vee (\wedge Y) = \wedge_{y \in Y}(x \vee y)$ (for finite lattices, infinite sup-distributivity amounts to distributivity), then, for each $c \in C$, $\phi_c \in uco^*(C)$. Even more, by this hypothesis on C, each ϕ_c is already co-additive. However, the following example shows that there exist complete lattices C which are not infinitely sup-distributive, such that each ϕ_c is meet-uniform, and for some $\hat{c} \in C$, $\phi_{\hat{c}}$ is not co-additive.

Example 4.1 Consider the following non-distributive lattice C.



It is simple to verify that, actually, for each $c \in C$, $\phi_c \in uco^*(C)$. Further, consider the closure $\phi_d \in uco(C)$, i.e. $\phi_d(C) = \{d, a, \top\}$. It is easily seen that ϕ_d is not co-additive: in fact, $\phi_d(b \wedge e) = \phi_d(\bot) = d$, while $\phi_d(b) \wedge \phi_d(e) = a \wedge \top = a$. □

Therefore, the class of complete lattices C such that, for any $c \in C$, ϕ_c is meet-uniform, is strictly wider than the class of infinitely sup-distributive lattices.

4.2 Downward Closures are Meet-Uniform

In this subsection, we show that the operator defined on the powerset of any poset satisfying the ascending chain condition which gives the *downward closure* of any subset is a meet-uniform closure. The family of these closures is of noticeable importance in theoretical computer science, notably in semantics of programming languages ([9]) and in models for polymorphic types ([10]).

Let $\langle A, \leq \rangle$ be any poset. Consider the following *downward* operator $\downarrow$ defined on the powerset $\wp(A)$ ordered by set-theoretic inclusion:

$$\downarrow S = \{x \in A \mid \exists y \in S.\, x \leq y\}.$$

Moreover, consider the operator $max : \wp(A) \to \wp(A)$ giving the *maximal elements* of any subset of A, and defined as follows:

$$max(S) = \{x \in S \mid \forall y \in S.\, x \leq y \Rightarrow x = y\}.$$

It is immediate to verify that max is idempotent and reductive on $\wp(A)_\subseteq$. However, max is neither monotonic nor anti-monotonic. In fact, consider the lattice C of Example 3.4. On the one hand, $\{a, b\} \subseteq \{a, b, \top\}$ whereas $max(\{a, b\}) = \{a, b\} \not\subseteq max(\{a, b, \top\}) = \{\top\}$. On the other hand, $\{a\} \subseteq \{a, b\}$ while $max(\{a\}) = \{a\} \not\supseteq max(\{a, b\}) = \{a, b\}$.

To show meet-uniformity of $\downarrow$, we need three preliminary lemmata.

Lemma 4.2 *For any $S \in \wp(A)$, $max(S) = max(\downarrow S)$.*

Proof. ($\subseteq$) Consider any $x \in max(S)$. We want to prove that $x \in max(\downarrow S)$. Thus, consider any $y \in \downarrow S$ such that $x \leq y$. There exists $z \in S$ such that $y \leq z$. Hence, $x \leq z$, from which $x = z$. Therefore, $x = y$, and since $x \in \downarrow S$, we get $x \in max(\downarrow S)$.
($\supseteq$) Consider any $x \in max(\downarrow S)$. We want to verify that $x \in max(S)$. If $x \in S$ then $x \in max(\downarrow S)$, since $S \subseteq \downarrow S$. Otherwise, $x \in \downarrow S \setminus S$. Then, there exists $y \in S$ such that $x < y$, namely we get a contradiction, since $x \in max(\downarrow S)$. ■

Lemma 4.3 *If A satisfies the ascending chain condition then for all $S \subseteq A$, $\downarrow S = \downarrow max(S)$.*

Proof. ($\supseteq$) Clear, since from $max(S) \subseteq S$ we directly get $\downarrow max(S) \subseteq \downarrow S$.
($\subseteq$) Consider any $x \in \downarrow S$. There exists $y_0 \in S$ such that $x \leq y_0$. If $y_0 \in max(S)$ then $x \in \downarrow max(S)$. Otherwise, there exists $y_1 \in S$ such that $y_0 \leq y_1$ and $y_0 \neq y_1$, viz. $y_0 < y_1$. If $y_1 \in max(S)$ then $x \in \downarrow max(S)$, otherwise, as before, we pick out another $y_2 \in S$ such that $y_0 < y_1 < y_2$. Iterating this constructive process, we should arrive to an infinite chain $\{y_i\}_{i \in \mathbb{N}} \subseteq A$ strictly increasing, viz. a contradiction. Thus, there exists $k \in \mathbb{N}$ such that $y_k \in max(S)$ and $x \leq y_k$, i.e. $x \in \downarrow max(S)$. ■

Lemma 4.4 *If A satisfies the ascending chain condition then for all $S, T \subseteq A$, $\downarrow S = \downarrow T \Leftrightarrow$ $max(S) = max(T)$.*

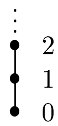
Proof. ($\Rightarrow$) $max(S) =$ (Lemma 4.2) $= max(\downarrow S) = max(\downarrow T) =$ (Lemma 4.2) $= max(T)$.
($\Leftarrow$) $\downarrow S =$ (Lemma 4.3) $= \downarrow max(S) = \downarrow max(T) =$ (Lemma 4.3) $= \downarrow T$. ■

Proposition 4.5 *If A satisfies the ascending chain condition then $\downarrow \in uco^*(\wp(A)_{\subseteq})$.*

Proof. Clearly $\downarrow \in uco(\wp(A)_{\subseteq})$. Thus, let us show that it is meet-uniform. Consider any family $\{S_i\}_{i\in I} \subseteq \wp(A)$ (where $|I| > 1$) such that $\exists k \in I. \forall i \in I.\ \downarrow S_i = \downarrow S_k$. We want to prove that $\downarrow(\cap_{i\in I} S_i) = \downarrow S_k$.
($\subseteq$) Since $\downarrow$ is monotonic.
($\supseteq$) Consider any $x \in \downarrow S_k$. By Lemma 4.3, $\downarrow S_k = \downarrow max(S_k)$, and so, there exists $y \in max(S_k)$ such that $x \leq y$. By Lemma 4.4, for all $i \in I$, $max(S_i) = max(S_k)$, and therefore, $max(S_k) \subseteq S_i$. Hence, $max(S_k) \subseteq \cap_{i\in I} S_i$. Thus, $y \in \cap_{i\in I} S_i$, i.e. $x \in \downarrow(\cap_{i\in I} S_i)$. ■

In the above proposition, the fact that A satisfies the ascending chain condition plays a fundamental rôle, as shown in the following example.

Example 4.6 Consider the poset $\mathbb{N}$ of the natural numbers, equipped with the standard ordering, depicted below, which does not satisfy the ascending chain condition.



Consider the subsets of odd and even numbers, O and E, respectively. Then, clearly, $\downarrow O = \downarrow E = \mathbb{N}$, while $\downarrow(O \cap E) = \downarrow\emptyset = \emptyset$, namely $\downarrow$ is not meet-uniform. □

Also, the following observation is interesting.

Remark 4.7 *$\downarrow$ is an additive closure.*

Proof. Consider any family $\{S_i\}_{i\in I} \subseteq \wp(A)$. We want to verify that $\downarrow(\cup_{i\in I} S_i) = \cup_{i\in I} \downarrow S_i$.
($\supseteq$) By monotonicity of $\downarrow$.
($\subseteq$) Consider any $x \in \downarrow(\cup_{i\in I} S_i)$. Then there exists $j \in I$ and $y \in S_j$ such that $x \leq y$. Hence, $x \in \downarrow S_j$, viz. $x \in \cup_{i\in I} \downarrow S_i$. ■

It is worth to observe that $\downarrow$ is not co-additive: considering the lattice C of Example 3.4, we have that $\downarrow(\{a\}\cap\{b\}) = \downarrow\emptyset = \emptyset$, while $\downarrow\{a\} \cap \downarrow\{b\} = \{\bot\}$. As a consequence of the above remark, if we consider the image, denoted by $\wp^{\downarrow}(A)$, of $\wp(A)$ by $\downarrow$, viz. $\wp^{\downarrow}(A) = \{S \in \wp(A) \mid S = \downarrow S\}$, we have that $\langle \wp^{\downarrow}(A), \subseteq\rangle$ is a complete sublattice of $\langle \wp(A), \subseteq\rangle$. Hence, the *lub* and the *glb* in $\wp^{\downarrow}(A)$ are just union and intersection, respectively. It is worth noting that $\wp^{\downarrow}(A)$ is also known as the *order ideal completion* of A (cf. [5]). Cousot and Cousot observed in [5] that the order ideal completion $\wp^{\downarrow}(A)$ coincides with the *Scott closed ideal completion*, or, equivalently, with the *lower* (or *Hoare*) *powerdomain* ([9]), whenever A satisfies the ascending chain condition, as in our case.

For the downward closure considered in this section, there is a simple expression of the canonical representative of Definition 3.6, as stated below.

Remark 4.8 *If A satisfies the ascending chain condition then, for each $I \in \wp(A)$, $\nabla(I) = max(I)$.*

Proof.

$$
\begin{array}{rcll}
\nabla(I) & = & \cap\{J \in \wp(A) \mid \downarrow J = \downarrow I\} & \\
& = & \cap\{J \in \wp(A) \mid max(J) = max(I)\} & \text{(Lemma 4.4)} \\
& = & max(I) & \text{(since } max \text{ is idempotent and reductive).}
\end{array}
$$
■

Finally, it is worthwhile to remark that "dualizing" this subsection, we get that whenever A satisfies the descending chain condition the *upward closure* is meet-uniform as well.

5 Lifting Complete Orders via Meet-Uniform Closures

Suppose $\langle C, \leq_C \rangle$ and $\langle D, \leq_D \rangle$ are posets, and consider any function $f : C \to D$.

Definition 5.1 The *lifting via* f of the ordering $\leq_C$ is the relation $\leq_C^f$ on C defined as follows:

$$x \leq_C^f y \quad \text{iff} \quad f(x) \leq_D f(y) \;\; \& \;\; (f(y) \leq_D f(x) \Rightarrow x \leq_C y). \qquad \square$$

The insight for this definition was provided by the paper [7], where an instance of $\leq_C^f$ was introduced to relate Herbrand models with more concrete proof-theoretic notions like *computed answers* and *correct answers* in Horn-clause first-order logic. Definition 5.1 generalizes that order to arbitrary closure operators, yet providing the basis to achieve co-additivity of meet-uniform closure operators.

It is straightforward to prove the correctness of the above definition.

Lemma 5.2 *$\leq_C^f$ is a partial order on C.*

Suppose that the function f is monotonic. Then, note that $\leq_C \subseteq \leq_C^f$, i.e. for all $x, y \in C$, $x \leq_C y \Rightarrow x \leq_C^f y$. Also, if C is bounded, with top and bottom elements (w.r.t. $\leq_C$) $\top$ and $\bot$, then C is bounded for the lifted order too, where $\top$ and $\bot$ are still the top and the bottom w.r.t. $\leq_C^f$.

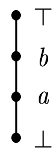
In the following, we will be interested in the case where f is a closure operator. For the moment, assume that $\langle C, \leq \rangle$ is a poset.

Proposition 5.3 *If $\rho \in uco(C_\leq)$ then $\rho \in uco(C_{\leq^\rho})$.*

Proof. Idempotency of ρ is clearly preserved. We prove monotonicity: $x \leq^\rho y \Rightarrow \rho(x) \leq^\rho \rho(y)$. $\rho(\rho(x)) \leq \rho(\rho(y))$ holds by hypothesis and idempotency of ρ. $\rho(\rho(y)) \leq \rho(\rho(x)) \Rightarrow x \leq y$ holds, since $\rho(\rho(y)) \leq \rho(\rho(x)) \Leftrightarrow \rho(y) \leq \rho(x) \Rightarrow x \leq y \Rightarrow \rho(x) \leq \rho(y)$. Finally, we show the extensivity of ρ w.r.t. $\leq^\rho$. $\rho(x) \leq \rho(\rho(x))$ holds by idempotency of ρ. $\rho(\rho(x)) \leq \rho(x) \Rightarrow x \leq \rho(x)$ holds by extensivity of ρ w.r.t. $\leq$. $\blacksquare$

Let us now give a simple example of lifting of a partial order via a closure operator.

Example 5.4 Reconsider the lattice $\langle C, \leq \rangle$ of Example 3.4, and recall the closures $\rho_2 = \{\top, a\}$ and $\rho_3 = \{\top, \bot\}$ defined on C. It is simple to verify that lifting $\leq$ via ρ_3 changes nothing in the structure of C, viz. $\leq \, = \, \leq^\rho$. On the other hand, lifting $\leq$ via ρ_2 provides the lattice depicted below.



$\square$

From now on, assume that $\langle C, \leq, \vee, \wedge, \top, \bot \rangle$ is a complete lattice. A relevant property of lifting a complete order $\leq$ via a meet-uniform closure operator is that it preserves the complete lattice structure. Given a meet-uniform closure $\rho \in uco^*(C)$, recall that for each $\bar{x} \in C$, the canonical representative of $\bar{x}$ is $\nabla(\bar{x}) = \wedge\{x \in C \mid \rho(x) = \rho(\bar{x})\}$. For any subset $Y \subseteq C$, consider the following definition of a possible *lub* $\dot{\vee}$ on the lifted poset $C_{\leq^\rho}$:

$$\dot{\vee} Y = \vee(\{y \in Y \mid \nabla(\vee Y) \leq y\} \cup \{\nabla(\vee Y)\}).$$

This is indeed a correct expression for the *lub* w.r.t. the lifted order $\leq^\rho$.

Lemma 5.5 *For all $Y \subseteq C$, $\rho(\dot{\vee} Y) = \rho(\vee Y)$.*

Proof.

$$\begin{array}{rcl}\rho(\dot{\vee}Y) & = & \rho(\vee(\{y \in Y \mid \nabla(\vee Y) \leq y\} \cup \{\nabla(\vee Y)\})) \\ & = & \rho(\vee(\{\rho(y) \in Y \mid \nabla(\vee Y) \leq y\} \cup \{\rho(\nabla(\vee Y))\})) \\ & = & \rho(\vee(\{\rho(y) \in Y \mid \nabla(\vee Y) \leq y\} \cup \{\rho(\vee Y)\})) \\ & = & \rho(\vee(\{y \in Y \mid \nabla(\vee Y) \leq y\} \cup \{\vee Y\})) \\ & = & \rho(\vee Y). \blacksquare\end{array}$$

Proposition 5.6 $\dot{\vee}$ *is the least upper bound in* $\langle C, \leq^\rho \rangle$.

Proof. Consider any $Y \subseteq C$.

(a) Let us show that if $y \in Y$ then $y \leq^\rho \dot{\vee}Y$.
(a_1) $\rho(y) \leq \rho(\dot{\vee}Y)$: Immediate from Lemma 5.5.
(a_2) $\rho(\dot{\vee}Y) \leq \rho(y) \Rightarrow y \leq \dot{\vee}Y$: Since, from the hypothesis and Lemma 5.5, $\rho(y) = \rho(\dot{\vee}Y) = \rho(\vee Y)$, we have that $\nabla(\vee Y) = \nabla(\rho(\vee Y)) = \nabla(\rho(y)) = \nabla(y)$, it is clear that this implies $y \leq \dot{\vee}Y$.

(b) Suppose that there exists $u \in C$ such that $\forall y \in Y.\ y \leq^\rho u$. In such a case, we want to prove that $\dot{\vee}Y \leq^\rho u$.
(b_1) $\rho(\dot{\vee}Y) \leq \rho(u)$: From the hypothesis follows that $\vee\rho(Y) \leq \rho(u)$. Hence, by Lemma 5.5, $\rho(\dot{\vee}Y) = \rho(\vee Y) = \rho(\vee\rho(Y)) \leq \rho(\rho(u)) = \rho(u)$.
(b_2) $\rho(u) \leq \rho(\dot{\vee}Y) \Rightarrow \dot{\vee}Y \leq u$: By hypothesis and Lemma 5.5, $\rho(u) = \rho(\vee Y)$. First, we verify that $\nabla(\vee Y) \leq u$: in fact, $\nabla(\vee Y) = \nabla(\rho(\vee Y)) = \nabla(\rho(u)) = \nabla(u) \leq u$. Second, if $\nabla(\vee Y) \leq y$ then $y \leq u$: as above, $\nabla(\vee Y) = \nabla(u)$, and therefore, $\nabla(u) \leq y$. Thus, $\rho(\nabla(u)) = \rho(u) \leq \rho(y)$. Since, by hypothesis, $\rho(u) \leq \rho(y) \Rightarrow y \leq u$, we get the desired $y \leq u$. $\blacksquare$

Notice that we added the singleton $\{\nabla(\vee Y)\}$ in the definition of $\dot{\vee}$ because if $\{y \in Y \mid \nabla(\vee Y) \leq y\}$ is empty then we take $\nabla(\vee Y)$ as the *lub*, rather than $\vee\emptyset = \bot$. As a corollary of the above proposition, we get the following important result.

Theorem 5.7 *If* $C_\leq$ *is a complete lattice and* $\rho \in uco^*(C_\leq)$ *then* $C_{\leq^\rho}$ *is a complete lattice.*

The *glb* $\dot{\wedge}$ in $C_{\leq^\rho}$ could be defined, as usual, as the least upper bound of the lower bounds, i.e. for each $Y \subseteq C$,

$$\dot{\wedge}Y = \dot{\vee}\{c \in C \mid \forall y \in Y.\ c \leq^\rho y\}.$$

For this *glb* the following is a useful alternative characterization. For any $Y \subseteq C$ define $\dot{\wedge}$ as follows:

$$\dot{\wedge}Y = \begin{cases} \wedge\{y \in Y \mid \rho(y) = \rho(\bar{x})\} & \text{if } \exists \bar{x} \in Y.\ \wedge\rho(Y) = \rho(\bar{x}); \\ \wedge\rho(Y) & \text{otherwise.}\end{cases}$$

Proposition 5.8 $\dot{\wedge}$ *is the greatest lower bound in* $\langle C, \leq^\rho \rangle$.

Proof. Consider any $Y \subseteq C$. We distinguish between the two mutually exclusive branches of the definition of $\dot{\wedge}$.

(a) There exists $\bar{x} \in Y$ such that $\wedge\rho(Y) = \rho(\bar{x})$, and hence, $\dot{\wedge}Y = \wedge\{y \in Y \mid \rho(y) = \rho(\bar{x})\}$. Since ρ is meet-uniform, $\rho(\dot{\wedge}Y) = \rho(\bar{x}) = \wedge\rho(Y)$. We will use this observation during the proof.
(a_1) $\forall y \in Y.\ \dot{\wedge}Y \leq^\rho y$: First, $\rho(\dot{\wedge}(Y)) = \rho(\bar{x}) \leq \rho(y)$ holds. Second, $\rho(y) \leq \rho(\dot{\wedge}Y) \Rightarrow \dot{\wedge}Y \leq y$ holds too, since the premise is equivalent to $\rho(y) = \rho(\bar{x})$.
(a_2) If there exists $l \in C$ such that $\forall y \in Y.\ l \leq^\rho y$ then $l \leq^\rho \dot{\wedge}Y$: First, $\rho(l) \leq \rho(\dot{\wedge}Y)$, since $\rho(\dot{\wedge}Y) = \rho(\bar{x})$, and, from the hypothesis, $\rho(l) \leq \wedge\rho(Y) = \rho(\bar{x})$. Second, the implication $\rho(\dot{\wedge}Y) \leq \rho(l) \Rightarrow l \leq \dot{\wedge}Y$ holds, since its premise implies $\rho(\bar{x}) = \rho(l)$, and, if $y \in Y$ is such that $\rho(y) = \rho(\bar{x})$ then $\rho(y) = \rho(l)$. By exploiting the hypothesis $l \leq^\rho y$, this implies $l \leq y$, which implies $l \leq \dot{\wedge}Y$.

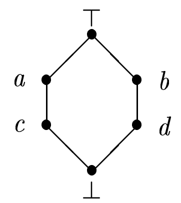
(b) Assume the other branch, i.e. $\forall y \in Y.\ \wedge\rho(Y) < \rho(y)$. In this case, $\dot{\wedge} Y = \wedge\rho(Y)$.
(b_1) $\forall y \in Y.\ \dot{\wedge} Y \leq^{\rho} y$: First, $\rho(\dot{\wedge} Y) = \rho(\wedge\rho(Y)) = \wedge\rho(Y) \leq \rho(y)$. Second, the implication $\rho(y) \leq \rho(\dot{\wedge} Y) \Rightarrow \dot{\wedge} Y \leq y$, viz. $\rho(y) = \wedge\rho(Y) \Rightarrow \dot{\wedge} Y \leq y$, trivially holds, since, by hypothesis, its premise is never satisfied.
(b_2) If there exists $l \in C$ such that $\forall y \in Y.\ l \leq^{\rho} y$ then $l \leq^{\rho} \dot{\wedge} Y$: First, $\rho(l) \leq \rho(\dot{\wedge} Y) = \wedge\rho(Y)$, since, by hypothesis, $\forall y \in Y.\ \rho(l) \leq \rho(y)$. Second, $\rho(\dot{\wedge} Y) \leq \rho(l) \Rightarrow l \leq \dot{\wedge} Y$ holds, since it is equivalent to $\rho(l) = \wedge\rho(Y) \Rightarrow l \leq \wedge\rho(Y)$, which is true, as $l \leq \rho(l)$. ■

Corollary 5.9 *For any $Y \subseteq C$, $\dot{\wedge}\rho(Y) = \wedge\rho(Y)$.*

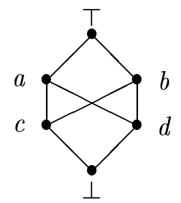
As observed in all generality at the beginning of this section, also note that $\top$ and $\bot$ remain unchanged after the lifting of the ordering.

It is important to note that the meet-uniformity property of ρ plays a fundamental rôle in Theorem 5.7. In fact, the following example shows that if the closure does not satisfy the meet-uniformity property then, in general, the lifted order does not give rise to a complete lattice.

Example 5.10 Consider the finite lattice $\langle C, \leq, \vee, \wedge, \top, \bot\rangle$ depicted below.



Consider also the closure $\rho \in uco(C)$ defined by $\rho(C) = \{\top, c, d, \bot\}$. It is immediate to verify that ρ is not meet-uniform: in fact, $\rho(a) = \rho(b) = \top$, but $\rho(a \wedge b) = \rho(\bot) = \bot$. In this case, the lack of meet-uniformity for ρ implies that the lifted poset $\langle C, \leq^{\rho}\rangle$ is not a lattice anymore. C equipped with the ordering $\leq^{\rho}$ is represented below.



Clearly, $\langle C, \leq^{\rho}\rangle$ is not a lattice, because, by definition of $\leq^{\rho}$, $d \leq^{\rho} a$ and $c \leq^{\rho} b$. □

Lifting the order $\leq$ via a meet-uniform closure ρ on the complete lattice $C_{\leq}$ is a process which preserves the meet-uniformity property, in addition to the property for ρ of being a closure (Proposition 5.3), as stated below.

Proposition 5.11 *If $\rho \in uco^*(C_{\leq})$ then $\rho \in uco^*(C_{\leq^{\rho}})$.*

Proof. Having shown Proposition 5.3, we have to prove meet-uniformity of ρ w.r.t. $\leq^{\rho}$ only. Thus, consider any $Y \subseteq C$ (with $|Y| > 1$), and assume that there exists $\bar{x} \in Y$ such that $\rho(y) = \rho(\bar{x})$, for all $y \in Y$. Then, since $\wedge\rho(Y) = \rho(\bar{x})$, $\dot{\wedge} Y = \wedge\{y \in Y \mid \rho(y) = \rho(\bar{x})\} = \wedge Y$. Hence, by meet-uniformity of ρ w.r.t. $\leq$, $\rho(\dot{\wedge} Y) = \rho(\wedge Y) = \rho(\bar{x})$. ■

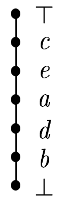
Lifting a complete order $\leq$ via a meet-uniform closure operator ρ has a remarkable consequence, which was the ultimate aim of this paper. In fact, by this process, we upgrade the properties enjoyed by ρ whenever considered w.r.t. the lifted complete order $\leq^{\rho}$: ρ becomes a co-additive closure operator.

Theorem 5.12 *If $\rho \in uco^*(C_{\leq})$ then ρ is co-additive on $\langle C, \leq^{\rho}\rangle$.*

Proof. By Corollary 5.9, it is enough to show that for any $Y \subseteq C$, $\rho(\dot{\wedge} Y) = \wedge\rho(Y)$. We distinguish the two branches of the characterization of $\dot{\wedge}$ given by Proposition 5.8.
(i) If $\dot{\wedge} Y = \wedge\{y \in Y \mid \rho(y) = \rho(\bar{x})\}$ for a suitable $\bar{x} \in Y$ such that $\rho(\bar{x}) = \wedge\rho(Y)$, then $\rho(\dot{\wedge} Y) = \rho(\bar{x}) = \wedge\rho(Y)$, because ρ is meet-uniform (on $C_{\leq}$).
(ii) Otherwise, $\dot{\wedge} Y = \wedge\rho(Y)$, and hence, $\rho(\dot{\wedge} Y) = \rho(\wedge\rho(Y)) = \wedge\rho(Y)$. ∎

We conclude this section showing the practical impact of the above theorem on the example of meet-uniform principal filters of Subsection 4.1.

Example 5.13 Let us recall Example 4.1. As shown there, the principal filter ϕ_d on C was meet-uniform but non-co-additive. Lifting the order of C via ϕ_d gives rise to the chain depicted below.



Now, it is simple to check that ϕ_d $(= \{d, a, \top\})$ has become co-additive with respect to this lifted order. □

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