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Abstract.

The uniqueness of the KKT multipliers of a nonlinear program has been studied in a well–known paper by Kyparisis. In the first part of this note, we show that the characterization obtained in that paper does not provide a satisfactory result for the multiobjective case. Thus, we introduce a new regularity condition, which involves also the objective functions, and we show that it is necessary and sufficient in order to have unique KKT multipliers. Moreover, we use this condition to refine a second order necessary optimality condition, which has been obtained in a recent paper.

Keywords: Multiobjective programming; Mangasarian–Fromovitz type conditions; second order optimality conditions.

1 Introduction

We consider the following constrained multiobjective program:

$$\min_{\operatorname{int}\mathbb{R}^{\ell}_{+}} f(x) \quad \text{subject to} \quad x \in X \tag{1}$$

where the feasible region is described by inequalities and equalities

$$X := \{\ x \in \mathbb{R}^n \ :\ g(x) \le 0, \quad h(x) = 0\ \}$$

with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^{\ell}$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ differentiable functions; let

$$K := \{1, ..., \ell\},\ I := \{1, ..., m\},\ J := \{1, ..., p\}$$

denote the index sets of the involved vector functions. Moreover, let

$$I(\overline{x}) := \{i \in I \ :\ g_i(\overline{x}) = 0\}$$

denote the index set of the active inequality constraints at a given $\overline{x} \in X$.

The notation $\min_{\operatorname{int}\mathbb{R}^{\ell}_{+}}$ marks vector minimum with respect to the cone $\operatorname{int}\mathbb{R}^{\ell}_{+}$: $\overline{x} \in X$ is said to be a *local vector minimum point* of (1) if and only if there exists a neighbourhood N of $\overline{x}$, such that no $x \in X \cap N$ satisfies $f(\overline{x}) - f(x) \in \operatorname{int}\mathbb{R}^{\ell}_{+}$, that is $f_k(\overline{x}) > f_k(x)$ for all $k \in K$.

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It is well–known (see [4]) that a necessary condition for $\overline{x} \in X$ to be a local vector minimum point of (1) is that there exist $\theta \in \mathbb{R}^\ell$, $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ not all zero such that

$$\sum_{k \in K} \theta_k \nabla f_k(\overline{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\overline{x}) + \sum_{j \in J} \mu_j \nabla h_j(\overline{x}) = 0, \tag{2}$$

$$\lambda_i g_i(\overline{x}) = 0, \quad i \in I, \tag{3}$$

$$\theta_k \geq 0, \quad k \in K, \quad \lambda_i \geq 0, \quad i \in I. \tag{4}$$

The vectors (θ, λ, μ) satisfying the above system are known as John multipliers; all the feasible points $\overline{x}$ for which John multipliers exist are called stationary.

In the case of scalar programming ($\ell = 1$), John multipliers are called KKT multipliers when $\theta = 1$. The structure of the set of KKT multipliers has been analysed by some authors (see for instance [3, 5, 7, 8, 9]). In particular, it has been proved in [3] that this set is nonempty and bounded if and only if the well–known Mangasarian-Fromovitz constraint qualification $(MFCQ)$ holds. To achieve uniqueness, the following modification of $(MFCQ)$ has been introduced in [5].

Definition 1.1 The *strict Mangasarian-Fromovitz constraint qualification* $(SMFCQ)$ holds at a stationary point $\overline{x} \in X$ when

- $\nabla g_i(\overline{x})$, $i \in I_+(\overline{x})$, $\nabla h_j(\overline{x})$, $j \in J$, are linearly independent;
- there exists $w \in \mathbb{R}^n$ such that

$$\begin{cases} \nabla g_i(\overline{x}) w & < & 0, & i \in I_o(\overline{x}), \\ \nabla g_i(\overline{x}) w & = & 0, & i \in I_+(\overline{x}), \\ \nabla h_j(\overline{x}) w & = & 0, & j \in J, \end{cases}$$

where the index sets

$$I_+(\overline{x}) := \{i \in I(\overline{x}) \ : \ \overline{\lambda}_i > 0\}, \quad I_o(\overline{x}) := I(\overline{x}) \setminus I_+(\overline{x})$$

depend upon the chosen multipliers.

Kyparisis [5] proved that the set of KKT multipliers is a singleton if and only if $(SMFCQ)$ holds.

In the multiobjective case θ is not a number but a vector; thus, a slightly different definition for KKT multipliers is needed. In order to avoid that two linearly dependent vectors of John multipliers are both KKT ones, it is straightforward to require that the norm of θ is fixed. Let

$$M_+(\overline{x}) := \{(\theta, \lambda, \mu) \ : \ \ (2), (3), (4) \text{ hold and } \|\theta\| = 1\}$$

be the set of KKT multipliers for problem (1), where $\| \cdot \|$ is any given norm in $\mathbb{R}^\ell$.

It would be reasonable to think that the result of [5] holds also in the multiobjective case. But, if we introduce the set

$$M_+(\overline{x}, \theta) := \{(\lambda, \mu) \ : \ (\theta, \lambda, \mu) \in M_+(\overline{x})\},$$

we can prove the following result just arguing as in [5].

Theorem 1.1 *Let $(\overline{\theta}, \overline{\lambda}, \overline{\mu}) \in M_+(\overline{x})$. Then, $M_+(\overline{x}, \overline{\theta})$ is a singleton if and only if $(SMFCQ)$ holds.*

Indeed, in the scalar case this theorem is exactly Proposition 1.1 of [5] and it provides a characterization of the uniqueness of the KKT multipliers; on the contrary, it is not a satisfactory result for the multiobjective case: there may be linearly independent unit vectors θ such that the corresponding sets $M_+(\overline{x}, \theta)$ are not empty, as the following example shows.

Example 1.1 Let us consider (1) with $n = 3,\ \ell = 2,\ m = 2,\ p = 1$ and

$$f_1(x_1, x_2, x_3) = x_1^2 + x_2, \quad f_2(x_1, x_2, x_3) = -x_1 - x_2 + x_3,$$

$$g_1(x_1, x_2, x_3) = x_1, \quad g_2(x_1, x_2, x_3) = -x_2,$$

$$h_1(x_1, x_2, x_3) = x_1^2 - x_3.$$

The vector $\overline{x} = (0, 0, 0)$ is a local vector minimum point of (1) and system (2)–(4) becomes

$$\theta_1 = \lambda_1 + \lambda_2, \quad \theta_2 = \lambda_1, \quad \theta_2 = \mu_1, \quad \lambda_1, \lambda_2 \geq 0.$$

Thus, easy calculations show that

$$M_+(\overline{x}) = \{(\theta^\alpha, \lambda^\alpha, \mu^\alpha)\ :\ \theta^\alpha = (\cos\alpha, \sin\alpha),\ \lambda^\alpha = (\alpha, \cos\alpha - \sin\alpha),\ \mu^\alpha = \alpha, \quad \alpha \in [0, \pi/4]\ \}.$$

Hence, there are not unique KKT multipliers. On the contrary, $M_+(\overline{x}, \theta^\alpha) = \{(\lambda^\alpha, \mu^\alpha)\}$ is a singleton for all $\alpha \in [0, \pi/4]$; in fact, it is straightforward to prove that $(SMFCQ)$ holds for all possible index sets $I_+(\overline{x})$.

2 Uniqueness of KKT multipliers

In order to gain uniqueness without fixing a unit vector θ a priori, we have to introduce a new regularity condition, strengthening $(SMFCQ)$ in the following way.

Definition 2.1 The *strict Mangasarian-Fromovitz regularity condition* $(SMFRC)_s$ holds at a stationary point $\overline{x}$ with respect to the index $s \in K$ when

- $\nabla f_k(\overline{x})$, $k \in K_+$ and $k \neq s$, $\nabla g_i(\overline{x})$, $i \in I_+(\overline{x})$, and $\nabla h_j(\overline{x})$, $j \in J$, are linearly independent;
- there exists $w_s \in \mathbb{R}^n$ such that

$$\begin{cases} \nabla f_k(\overline{x}) w_s & < & 0, & k \notin K_+,\ k \neq s, \\ \nabla g_i(\overline{x}) w_s & < & 0, & i \in I_o(\overline{x}), \\ \nabla f_k(\overline{x}) w_s & = & 0, & k \in K_+,\ k \neq s, \\ \nabla g_i(\overline{x}) w_s & = & 0, & i \in I_+(\overline{x}), \\ \nabla h_j(\overline{x}) w_s & = & 0, & j \in J, \end{cases} \tag{5}$$

where the index set

$$K_+ := \{k \in K \ : \ \overline{\theta}_k > 0\}$$

depends upon the chosen multipliers.

The above condition is a quite strong requirement; in fact, the solvability of system (5) forces some fixed components of all KKT multipliers to be zero, as the following lemma shows.

Lemma 2.1 *Let $(\overline{\theta}, \overline{\lambda}, \overline{\mu}) \in M_+(\overline{x})$ and suppose that (5) admits a solution for some $s \in K_+$. If $(\theta', \lambda', \mu') \in M_+(\overline{x})$, then*

$$\begin{aligned} \{k \in K \ : \ \theta'_k > 0\} \ &\subseteq \ K_+, \\ \{i \in I \ : \ \lambda'_i > 0\} \ &\subseteq \ I_+(\overline{x}). \end{aligned}$$

Proof. Let $w_s \in \mathbb{R}^n$ be a solution to (5) for some $s \in K_+$. Since $(\overline{\theta}, \overline{\lambda}, \overline{\mu}) \in M_+(\overline{x})$, we have

$$0 = \sum_{k \in K} \overline{\theta}_k \nabla f_k(\overline{x}) w_s + \sum_{i \in I} \overline{\lambda}_i \nabla g_i(\overline{x}) w_s + \sum_{j \in J} \overline{\mu}_j \nabla h_j(\overline{x}) w_s = \overline{\theta}_s \nabla f_s(\overline{x}) w_s.$$

Since $\overline{\theta}_s \neq 0$, we get $\nabla f_s(\overline{x}) w_s = 0$. Thus, w_s solves also the system

$$\begin{cases} \nabla f_k(\overline{x}) w_s & < & 0, & k \notin K_+, \\ \nabla g_i(\overline{x}) w_s & < & 0, & i \in I_o(\overline{x}), \\ \nabla f_k(\overline{x}) w_s & = & 0, & k \in K_+, \\ \nabla g_i(\overline{x}) w_s & = & 0, & i \in I_+(\overline{x}), \\ \nabla h_j(\overline{x}) w_s & = & 0, & j \in J. \end{cases}$$

By Motzkin Theorem of the Alternative (see [6]), system (2) has no solution (θ, λ, μ) such that $\theta_k \geq 0$ for all $k \notin K_+$, $\lambda_i \geq 0$ for all $i \in I_o(\overline{x})$ and at least one of them is nonzero. Since $(\theta', \lambda', \mu')$ satisfies (2), we have $\theta'_k = 0$ for all $k \notin K_+$ and $\lambda'_i = 0$ for all $i \notin I_+(\overline{x})$. □

Thus, we can prove the following characterization of the uniqueness of KKT multipliers for the multiobjective case.

Theorem 2.1 *Let $(\overline{\theta}, \overline{\lambda}, \overline{\mu}) \in M_+(\overline{x})$. Then,*

$M_+(\overline{x})$ is a singleton if and only if $(SMFRC)_s$ holds at $\overline{x}$ for all $s \in K_+$.

Proof. Given any $s \in K_+$, suppose that $(SMFRC)_s$ does not hold. Whether the linear independence condition does not hold or system (5) has no solution, there exist $\theta' \in \mathbb{R}^m$, $\lambda' \in \mathbb{R}^p$, $\mu' \in \mathbb{R}^q$ not all zero such that $\theta'_s = 0$, $\lambda'_i = 0$ for all $i \notin I(\overline{x})$ and

$$\sum_{k \in K} \theta'_k \nabla f_k(\overline{x}) + \sum_{i \in I} \lambda'_i \nabla g_i(\overline{x}) + \sum_{j \in J} \mu'_j \nabla h_j(\overline{x}) = 0.$$

Without any loss of generality we can assume that

$$\begin{aligned} \max\{|\theta'_k| \ : \ k \in K\} \ &< \ \min\{\overline{\theta}_k \ : \ k \in K_+\}, \\ \max\{|\lambda'_i| \ : \ i \in I_+(\overline{x})\} \ &< \ \min\{\overline{\lambda}_i \ : \ i \in I_+(\overline{x})\}. \end{aligned}$$

Then,

$$\begin{aligned} \theta'_k + \overline{\theta}_k \geq 0, \qquad & k \in K, \\ \lambda'_i + \overline{\lambda}_i \geq 0, \qquad & i \in I. \end{aligned}$$

Moreover, if we set $\gamma = \|\theta' + \overline{\theta}\|^{-1}$, we have that

$$\gamma(\theta' + \overline{\theta}, \lambda' + \overline{\lambda}, \mu' + \overline{\mu}) \in M_+(\overline{x}).$$

By the uniqueness assumption, we must have

$$\begin{aligned} \gamma\theta'_k &= (1-\gamma)\overline{\theta}_k, \qquad & k \in K, \\ \gamma\lambda'_i &= (1-\gamma)\overline{\lambda}_i, \qquad & i \in I, \\ \gamma\mu'_j &= (1-\gamma)\overline{\mu}_j, \qquad & j \in J. \end{aligned}$$

Since $\theta'_s = 0$ and $\overline{\theta}_s \neq 0$, it must be $\gamma = 1$; thus, we have $\theta' = 0,\ \lambda' = 0,\ \mu' = 0$ in contradiction with the assumption.

Vice versa, let $(\theta', \lambda', \mu')$ be any vector of $M_+(\overline{x})$ and $s \in K$ be such that $\theta'_s \neq 0$. By Lemma 2.1 we have that $\theta'_k = 0$ for all $k \notin K_+$ and $\lambda'_i = 0$ for all $i \notin I_+(\overline{x})$. Thus, $s \in K_+$ and we get

$$\sum_{k \in K_+} (\gamma\theta'_k - \overline{\theta}_k)\nabla f_k(\overline{x}) + \sum_{i \in I_+(\overline{x})} (\gamma\lambda'_i - \overline{\lambda}_i)\nabla g_i(\overline{x}) + \sum_{j \in J} (\gamma\mu'_j - \overline{\mu}_j)\nabla h_j(\overline{x}) = 0, \tag{6}$$

where $\gamma = \overline{\theta}_s/\theta'_s$. Since $\gamma\theta'_s - \overline{\theta}_s = 0$, then all the coefficients in (6) must be zero because of the linear independence condition of $(SMFRC)_s$. Thus, we have that $\gamma(\theta', \lambda', \mu') = (\overline{\theta}, \overline{\lambda}, \overline{\mu})$. Moreover,

$$1 = \|\overline{\theta}\| = \gamma\|\theta'\| = \gamma,$$

and then $(\theta', \lambda', \mu') = (\overline{\theta}, \overline{\lambda}, \overline{\mu})$, proving the uniqueness of the given KKT multipliers. □

In the scalar case, $K_+ = K$ consists just of the only index corresponding to the objective function and consequently $(SMFRC)$ reduces to $(SMFCQ)$; thus, also Theorem 2.1 collapses to Proposition 1.1 of [5].

Remark 2.1 The above characterization of uniqueness depends only upon the indexes in K_+; actually, $(SMFRC)_s$ can not hold whenever $s \notin K_+$. In fact, if $\overline{\theta}_s = 0$ then system (2) implies

$$\sum_{k \in K_+,\, k \neq s} \overline{\theta}_k \nabla f_k(\overline{x}) + \sum_{i \in I_+(\overline{x})} \overline{\lambda}_i \nabla g_i(\overline{x}) + \sum_{j \in J} \overline{\mu}_j \nabla h_j(\overline{x}) = 0$$

in contradiction with the linear independence condition.

Therefore, when $(SMFRC)_s$ holds for all $s \in K_+$ we can simply say that the strict Mangasarian-Fromovitz regularity condition $(SMFRC)$ holds, without any possible misunderstanding.

3 A second order necessary optimality condition

Since there may be stationary points which are not local vector minimum points of (1), additional second order optimality conditions can be developed when all the functions are twice differentiable. Actually, $(SMFRC)$ can be employed also as a second order regularity condition. The following John type necessary optimality condition has been proved in [2].

Theorem 3.1 *If $\overline{x} \in X$ is a local vector minimum point of (1), then for any descent direction v, that is any $v \in \mathbb{R}^n$ satisfying the following system*

$$\begin{cases} \nabla f_k(\overline{x})v & \leq & 0, & k \in K, \\ \nabla g_i(\overline{x})v & \leq & 0, & i \in I(\overline{x}), \\ \nabla h_j(\overline{x})v & = & 0, & j \in J, \end{cases}$$

there exist $\theta \in \mathbb{R}^\ell$, $\lambda \in \mathbb{R}^m$, and $\mu \in \mathbb{R}^p$ not all zero such that (2), (3) and (4) hold; moreover, the second order condition

$$\left(\sum_{k \in K} \theta_k \nabla^2 f_k(\overline{x}) + \sum_{i \in I} \lambda_i \nabla^2 g_i(\overline{x}) + \sum_{j \in I} \mu_j \nabla^2 h_j(\overline{x}) \right)(v, v) \geq 0, \tag{7}$$

and the first order complementarity slackness conditions

$$\theta_k \nabla f_k(\overline{x})v = 0, \quad k \in K, \tag{8}$$

$$\lambda_i \nabla g_i(\overline{x})v = 0, \quad i \in I(\overline{x}), \tag{9}$$

are satisfied.

In the above theorem no regularity assumption is required but θ can be zero. To avoid this drawback, some regularity conditions have been considered in [1, 2, 10]. However, even with these additional assumptions, the multipliers are not fixed but they may change with the considered descent direction. On the contrary, assuming $(SMFRC)$, not only θ is different from zero but also the multipliers do not depend upon the descent directions.

Theorem 3.2 *Let $\overline{x} \in X$ be a local vector minimum point of (1) and (θ, λ, μ) be such that (2), (3) and (4) hold. If $(SMFRC)$ holds at $\overline{x}$, then there exists $\gamma > 0$ such that $\gamma(\theta, \lambda, \mu) \in M_+(\overline{x})$ and (7), (8) and (9) hold for all descent directions.*

Proof. Suppose that $\theta = 0$; then, (2) implies that $\nabla g_i(\overline{x})$ for all $i \in I_+(\overline{x})$ and $\nabla h_j(\overline{x})$ for all $j \in J$ are linearly dependent in contradiction with $(SMFRC)$. Thus, $\theta \neq 0$ and choosing $\gamma = \|\theta\|^{-1}$, we get $\gamma(\theta, \lambda, \mu) \in M_+(\overline{x})$. The fact that (7), (8) and (9) hold for all descent directions follows immediately from Theorem 3.1 and Theorem 2.1. □

An analogous result has been obtained in [10] under weaker regularity conditions (Mc Cormick and Guignard constraint qualifications) but adding a convexity assumption, which allows to reduce problem (1) to a scalar nonlinear program. Thus, Theorem 3.4 of [10] is a better result than Theorem 3.2 for convex multiobjective programming; on the contrary, Theorem 3.2 can be applied also to nonconvex problems.

Example 3.1 Let us consider (1) with $n = 3,\ \ell = 3,\ m = 1,\ p = 1$ and

$$f_1(x_1, x_2, x_3) = -x_1, \quad f_2(x_1, x_2, x_3) = x_1 - x_3, \quad f_3(x_1, x_2, x_3) = x_2 - 2x_1^2,$$
$$g_1(x_1, x_2, x_3) = -x_1 - x_1^2, \quad h_1(x_1, x_2, x_3) = x_1^2 - x_2 + x_3^2.$$

The vector $\overline{x} = (0, 0, 0)$ is a local vector minimum point of (1) and system (2)–(4) becomes

$$\theta_1 - \theta_2 + \lambda_1 = 0, \quad \theta_3 = \mu_1, \quad \theta_2 = 0, \quad \theta_1, \theta_3, \lambda_1 \geq 0.$$

Then, there are unique KKT multipliers; in fact, we have that

$$M_+(\overline{x}) = \{(0, 0, 1, 0, 1)\}$$

and $(SMFRC)$ holds; since all the assumptions are satisfied, Theorem 3.2 can be applied. On the contrary, it is impossible to reduce problem (1) to a scalarized nonlinear program, having $\overline{x}$ as a minimum point. By Lemma 2.2 of [10] this implies that the convexity assumption of Theorem 3.4 of [10] is not satisfied.

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