Distribution network design: new problems and related models

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February 21, 2001

Abstract

We study some complex distribution network design problems, which involve facility location, warehousing, transportation and inventory decisions. Several realistic scenarios are investigated.

Two kinds of mathematical programming formulations are proposed for all the introduced problems, together with a proof of their correctness. Some formulations extend models proposed by Perl and Daskin (1985) for some warehouse location-routing problems; other formulations are based on flow variables and constraints.

1 Introduction

All companies that aim to be competitive on the market have to pay attention to their organization related to the entire supply chain. In particular, companies have to analyze the supply chain in order to improve the customer service level without an uncontrolled growth of costs. In few words, companies have to increase the efficiency of their logistics operations. It is therefore of fundamental importance to optimize the flows of goods (and also of information) among the actors of the supply chain, that is suppliers, producers, distributors, and customers.

The Efficient Customer Response (ECR) study [3] pointed out that vertical as well as horizontal integrations are required for the optimization of the flows in the supply chain, and for the optimization of all related activities. That implies agreements among subjects that operate at different levels of the supply chain (vertical integration), and among actors of the same level (horizontal integration). Clearly, before looking for external agreements, each actor should increase the efficiency in its internal organization.

In this work, we focus on some internal organization issues of the supply chain; in particular, we analyze some network design problems which are typical of producer companies (although they are significant for other types of

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companies, like the distribution companies). The aim is to optimize the flows of goods through the producer network, also called in the literature *distribution network*, from the plants where the goods are produced (supply points) to the demand points, which are essentially distributors such as wholesalers or retailers.

While analyzing a distribution network, we can generally distinguish between:

• the optimization of the flows of goods: in this case we consider an existing distribution network, made up of several layers (plants, central depots, regional depots, transit points and customers), and we want to optimize the flows of goods from the plants to the demand points, through the existing network, in order to satisfy the goals of the company while minimizing the transportation costs;

• the evaluation of the existing network and the solution of some network design problems, in order to improve the existing network: in this case we want to choose the best configuration of the facilities in the network in order to satisfy the goals of the company, while minimizing the overall costs.

Distribution network design problems involve both kinds of analysis. More precisely, these problems consist in determining the best way to transfer goods from the supply to the demand points by choosing the structure of the network (layers, different kinds of facilities operating at different layers, their number and their location), while minimizing the overall costs.

Distribution network design problems involve strategic decisions which influence tactical and operational decisions [2]. In particular, these problems involve facility location, transportation and inventory decisions, which affect the cost of the distribution system and the quality of the customer service level. So, they are core problems for each company.

For example, location decisions, such as the number and the location of facilities, the assignment of facilities to supply sources, and the allocation of demand to facilities, affect inventory decisions, such as the total system inventory and the location of inventories. Vice versa, inventory decisions affect location ones.

Inventory decisions also affect transportation decisions, and, vice versa, transportation decisions, such as the mode and the type of carriage, influence inventory decisions.

Finally, facility location decisions affect transportation decisions and vice versa. For instance, when the transportation cost is slowly decreasing with the volume and fast increasing with the distance, it may influence the number of facilities to locate.

As emerged from the above discussion, distribution network design problems involve a lot of integrated decisions, which are difficult to consider all together when solving the problems. Generally, some simplifying assumptions have been adopted in the literature for dealing with distribution network design problems, and only some aspects related to the complex network decisions have been modellized. For instance, in the past some authors dealt with distribution network design problems as pure location problems, without trying to address and integrate the different types of strategic decisions.

Webb [21] and, more recently, Salhi and Rand [18] recognized the error introduced into location problems by ignoring the interdependence between routing and location decisions. Since then, some papers focused on the relationships between facilities and transportation costs, stressing that location of distribution facilities and routing of vehicles from facilities are interdependent decisions. In particular, in recent years, some location-routing problems (LRP) arising in the context of distribution network design problems have been investigated. In these problems, the facility location and the vehicle routing aspects are solved simultaneously [6]. Given a set of candidate depot sites and customer requirements, in its simplest form LRP consists in determining the location of the depots and the routes of the vehicles for serving the customers, in such a way that some constraints, generally related to depot and vehicle capacity, route lengths and durations, and all the customer requirements are satisfied, while minimizing an objective function involving routing costs, vehicle fixed costs, depot fixed costs and depot operating costs. A review of the main literature devoted to LRP in the context of distribution network design problems is presented in the next section, together with a formal definition of the problem.

The distribution networks which have been addressed in LRP problems are quite simple, since in general they involve only two layers: depots and customers. Furthermore, no LRP studied in the literature includes inventories, except for two cases. In [17], a mathematical model for explicitly representing the trade off among facility, transportation and inventory costs is proposed; this integrated model differs from existing models only in the form of the objective function. In [13], the authors try to estimate the inventory costs and include them in the fixed costs related to facilities.

In this work, we will address more complex distribution network design problems, which involve facility location, transportation and inventory decisions. We will refer to these problems as the *integrated distribution network design problems.* More precisely, we will consider distribution networks made up of four layers (plants, central depots, regional depots and customers (demand points)), with the aim of defining the number and the location of the different types of facilities for improving the existing network. The analysis will take into account facility, warehousing, transportation and inventory costs. Several realistic scenarios will be investigated.

Dynamic versions of the integrated problems will be discussed as well, where the distribution network is analyzed over a certain time horizon rather than in a fixed period of time.

The main contribution of this paper is the statement of two kinds of mathematical programming formulations for all the introduced problems, together with a proof of their correctness. Some formulations extend models proposed in [16] for some warehouse location-routing problems; other formulations are based on flow variables and constraints. These formulations have to be viewed as the first step towards the definition of solution methods for complex distribution network design problems. The research is under development, and it will be the subject of a future work.

The paper is organized as follows. Section 2 contains a review of the main location-routing problems proposed in the literature to model important aspects of distribution network design problems, while Section 3 reviews the warehouse location-routing model proposed in [16]. In Section 4, we introduce the integrated distribution network design problems under investigation. Sections 5 and 6 present models to mathematically formulate the problems introduced in Section 4. Finally, Section 7 contains some conclusions and future research development.

2 The literature review

In [6], the distribution network design problems are classified according to the number of the layers in the distribution network, and to the type of routes between layers. In particular, Laporte introduced the terminology 'route of type R' (for *replenishment*), if the route connects a pair of nodes of two different layers (for instance, a depot is connected to a customer), and 'route of type T' (for *tour*) if it is a tour connecting a node in a layer with more nodes belonging to other layers (for instance, a depot is connected via a tour to a certain number of customers served by the same vehicle). Laporte observed that a distribution network design problem can be formulated as a location-routing problem (LRP) if and only if routes of type T are allowed, and location decisions arise at least at one layer. Otherwise, the problem reduces to a pure location or to a pure vehicle routing problem, which are two important components of LRP, very difficult on their own.

In the last two decades, many LRP models have been proposed in the literature to formulate and solve distribution network design problems. Most of them are related to a simple network with two layers (depots and customers), with routes of type T allowed between these layers. Each model differs from others for the number of depots to locate (single depot or multi-depot), for the presence of capacity constraints (depot capacity and vehicle capacity) and other route constraints, and for the form of the objective function.

In [6] some mathematical models are proposed by distinguishing between three-index and two-index formulations. Two-index formulations were used for: the single depot LRP, solved in [7] via an exact approach; the multi-depot LRP, which is an extension of the single depot LRP [9]; the multi-depot capacitated LRP [8], solved in an exact way; and some asymmetric versions [10].

Some interesting three-index formulations were proposed in [14] for solving the problem of locating regional blood banks to serve hospitals, and in [16] for evaluating the design of the division's distribution system, with regard to the number, size, and locations of central depots, in the area of Missouri, Oklahoma and Western Kansas. Hansen et al. [4] modified the integer linear programming formulation of Perl and Daskin [16] in order to provide an improved formulation, based on flow variables and flow constraints, and then they used the model for helping a Danish company operating in the chemical industry to choose the plant where to start a new production, and also as a long term strategical decision tool in a regional study for evaluating the economical effects of a fixed link between the two main Danish islands.

Another classification of location-routing problems can be found in [12].

As far as solution methods are concerned, due to the complexity of LRP, exact methods have been limited to small size instances, and to two-index formulations. Three-index formulations, more versatile but more heavy, were not solved exactly until now.

For solving larger problems and real instances, the only helpful methods have been heuristics (see, for example, [5] and [11], where the practical use of LRP for designing a newspaper distribution system is illustrated). The major part of the heuristic approaches are based on the decomposition of the problem into subproblems which are then solved sequentially, in order to address interdependencies. Also subproblems are usually solved in an approximate way ([16],[4]). Another popular approach, also used within the decomposition methods, is the saving method ([19], [4]).

A new approach is presented in [20]. The authors proposed a two-phase tabù search approach which integrates facility location and routing decisions. They also compared the performance of alternative LRP heuristics, by comparing their approach with one of the algorithms proposed in [19], and by furnishing a set of test problems.

Finally, we want to mention an interesting set-partitioning formulation of some LRP problems proposed in [1]. The idea is related to the successful implementation of exact algorithms for routing problems based on set partitioning models, and to the fact that many types of transportation problems can be formulated as set partitioning problems. The author developed a column generation heuristic and a partial branch-and-price algorithm, by solving to optimality instances with 25 facilities and 150 customers.

3 The warehouse location-routing problem

In [15] and [16] the warehouse location-routing problem (WLRP) was defined as the problem of a company which has to ship goods from some supply sources (plants) to a certain number of depots via truck loads, and then has to deliver the goods from the depots to some customers. More precisely, we are given the location of p plants, the location and the requirement of a set of n customers (specified in units of a single representative commodity), and a set of potential sites for the location of central depots. The warehouse location-routing problem consists in determining the number, the size and the location of the central depots, the allocation of the n customers to the depots, the direct replenishment from the plants to the depots (routes of type R), and the delivery tours for serving the customers from the central depots, so as to minimize the sum of the transportation costs and the warehousing costs, and by satisfying the following constraints:

1) each customer must be assigned to exactly one depot;

2) the requirements of the customers must be satisfied, and each customer must be served with only one vehicle (tour);

3) some capacity constraints, associated with depots and vehicles, must be satisfied, together with other route constraints, like a limit to the number of routes available for serving the customers and to their length.

Perl and Daskin [16] proposed a three-index formulation of WLRP, which is based on the following notation. Denote by:
$P = \{1, \ldots, p\}$ the set of plants;
$C = \{1, \ldots, n\}$ the set of customers;
$W = \{1, \ldots, m\}$ the set of potential depots;
$V = \{1, \ldots, v\}$ the set of vehicles.
Introduce a directed graph $G = (N, A)$, where:
$N = C \cup W \cup P$ is the set of the nodes;
$Y = C \cup W$ is the set of the customers and of the potential depots, $Y \subset N$;
A is the set of the arcs (representing the available links between the nodes of G).
Furthermore, define:
D_i : demand of customer $i, \forall i \in C$;
WC_j : capacity (in the sense of maximum throughput) of depot $j, \forall j \in W$;
C_{gh} : distance (in kilometers) between the nodes g and $h, \forall g, h \in Y$;
K_k : capacity of vehicle $k, \forall k \in V$;
Dm_k : maximum allowable length of the route of vehicle $k, \forall k \in V$;
VC_j : warehousing cost per unit throughput at depot $j, \forall j \in W$;
FC_j : fixed cost of establishing depot $j, \forall j \in W$;
CP_{uj} : unit transportation cost from the plant u to the depot $j, \forall u \in P, \forall j \in W$;
DC_k : transportation cost per kilometer of vehicle $k, \forall k \in V$.

We can now state the formulation, which is based on the following set of variables:

$$x_{ghk} = \begin{cases} 1 & if g precedes h in the route of vehicle k, \forall g, h \in Y, \forall k \in V \\ 0 & otherwise \end{cases}$$

$$y_{ij} = \begin{cases} 1 & if customer i is allocated to depot j, \forall i \in C, \forall j \in W \\ 0 & otherwise \end{cases}$$

$$z_j = \begin{cases} 1 & if depot j is opened, \forall j \in W \\ 0 & otherwise \end{cases}$$

f_{uj}: quantity shipped from the plant u to the depot $j, \forall u \in P, \forall j \in W$.

$$(WLRP) \qquad Min \sum_{j \in W} FC_j * z_j + \sum_{j \in W} VC_j \left(\sum_{i \in C} D_i * y_{ij} \right) \tag{1}$$
$$+ \sum_{k \in V} \sum_{g \in Y} \sum_{h \in Y} C_{gh} * DC_k * x_{ghk} + \sum_{u \in P} \sum_{j \in W} CP_{uj} * f_{uj}$$

$$
\begin{aligned}
\sum_{k \in V} \sum_{h \in Y} x_{ihk} &= 1 \qquad \forall i \in C && (2)\\
\sum_{i \in C} D_i \sum_{h \in Y} x_{ihk} - K_k &\leq 0 \qquad \forall k \in V && (3)\\
\sum_{g \in Y} \sum_{h \in Y} C_{gh} * x_{ghk} &\leq Dm_k \qquad \forall k \in V && (4)\\
\sum_{u \in P} f_{uj} - WC_j * z_j &\leq 0 \qquad \forall j \in W && (5)\\
\sum_{u \in P} f_{uj} - \sum_{i \in C} D_i * y_{ij} &= 0 \qquad \forall j \in W && (6)\\
\sum_{g \in Y} x_{hgk} - \sum_{g \in Y} x_{ghk} &= 0 \qquad \forall h \in Y, \forall k \in V && (7)\\
\sum_{g \in S} \sum_{h \in \bar{S}} \sum_{k \in V} x_{ghk} &\geq 1 \qquad \forall S \subset Y, t.c. W \subset S && (8)\\
\sum_{i \in C} \sum_{j \in W} x_{ijk} &\leq 1 \qquad \forall k \in V && (9)\\
\sum_{h \in Y} x_{ihk} + \sum_{h \in Y} x_{jhk} - y_{ij} &\leq 1 \qquad \forall i \in C, \forall j \in W, \forall k \in V && (10)\\
x_{ghk} &\in \{0,1\}, \forall g, h \in Y, \forall k \in V && (11)\\
y_{ij} &\in \{0,1\}, \forall i \in C, \forall j \in W\\
z_j &\in \{0,1\}, \forall j \in W\\
f_{uj} &\geq 0, \forall u \in P, \forall j \in W
\end{aligned}
$$

The objective function minimizes the sum of the fixed and the warehousing costs, plus the transportation costs. Constraints (2) impose that each customer be visited by a single vehicle (therefore, the demand of each customer is assumed to be less than the capacity of each vehicle). Constraints (3) are the vehicle capacity constraints, while (4) are related to the maximum length of the routes.

Constraints (5) and (6) are related to the flow of goods into the depots; in particular, constraints (5) limit the flow through a depot to its capacity, while constraints (6) require that the flow out of a depot be equal to the flow into it.

Constraints (7) oblige each vehicle which enters a node to leave the same node.

Constraints (8) and (9) specify the kinds of routes allowed in the solution; in particular, constraints (8) impose that each delivery route be connected to at least one depot (subtours among customers are not allowed), while constraints (9) imply that, in the optimal solution, only one depot is included in each route.

Finally, constraints (10) link the allocation and the routing components of WLRP: a customer i is assigned to a depot j if the vehicle k which visits customer i starts its trip from depot j.

4 The integrated distribution network design problems

Here we introduce some distribution network design problems involving several kinds of nodes at different layers of the distribution network, and more possibilities for supplying depots and for serving customers. According to Laporte's classification [6], in their general form the problems can be categorized as 4/R/T/T, that means distribution networks made up of 4 layers, with routes of type replenishment and of type tour. The goal of the analysis is to determine the best distribution system in order to minimize facility, warehousing, transportation and inventory costs, and to grant a certain customer service level.

Let us start by defining the nodes of the distribution network, and the kinds of available routes.

The nodes of the distribution network are partitioned into *plants* (or supply points, P), *central depots* (CD), *regional depots* (RD), *transit points* (TP), *clients* (C) and *big clients* (BC). The nodes are grouped into four layers as follows: supply points (P), central depots (CD), regional facilities (RD and TP) and demand points (BC and C).

In the network, plants send goods to CD; CD transfer them to TP and RD; TP and RD serve the set of clients C, while BC are served directly by CD because of their large demand. TP are similar to RD, but they do not maintain inventories; in fact, TP receive goods and immediately deliver them to the clients.

We assume to know the location and the demand of each client and big client, whose demands are specified in units of a single representative commodity. Also the location and the capacity of each potential facility are known. In particular, the capacity of CD and RD is the maximum inventory level that they are able to maintain, while the capacity of a transit point is the maximum quantity of commodity that may enter the transit point. Furthermore, we assume to know the maximum number of available vehicles, together with the capacity of each vehicle. The fleet of vehicles is non homogeneous, and it is common to the different layers of the distribution network.

Finally, as far as the connection among the nodes of the network is concerned, we admit the following kinds of routes: direct replenishment from the plants to the CD (routes of type R); tours from CD to RD and TP (including exactly one CD); delivery tours from CD to BC (including exactly one CD); mixed tours from CD to RD, TP and BC (including exactly one CD); and delivery tours from CD, RD or TP to the clients C (including exactly one facility among CD, RD and TP).

We can now define the integrated distribution network design problem, in its general form, as the problem of determining:

• *(location decisions)* where to locate CD, RD, TP, and the number of each type of facility;

• *(allocation decisions)* how to allocate the clients to the open facilities (CD, RD or TP); how to allocate the big clients to the open CD; how to allocate the open RD and TP to the open CD for their supply; how to allocate the open CD to the plants;

• *(routing decisions)* the vehicle routes for serving the clients, starting from a CD, a RD or a TP (we will refer to this routing as *routing of the second level*), and the mixed tours for visiting RD, TP and BC, starting from a CD (we will refer to this routing as *routing of the first level*), in such a way each client and big client, as well as each open facility (in the routing of the first level), be served by exactly one vehicle, and by satisfying capacity constraints related to the vehicles;

• *(inventory decisions)* the quantity of good which must be shipped from the plants to the CD, and from the CD to the RD and the TP, and the inventory level at CD and RD, in such a way to satisfy the capacity constraints at CD and TP;

in order to guarantee a certain customer service level, and to minimize the sum of the facility, warehousing, transportation and inventory costs. The customer service level is expressed as a minimum stock level which has to be maintained at each open CD and RD.

The introduced problem involves many strategic decisions which arise in designing distribution networks of producer companies. It is really complex for several reasons. One is that the location decisions are related to two different kinds of facilities, which have a different role in the distribution network, that is CD from one hand, and RD and TP from the other hand. Their location influences all the available routes.

Observe that the problem under investigation differs from WLRP, reviewed in Section 3, in many aspects, like:

• there are different types of facilities to locate and, consequently, the distribution network is more complex;

• there are different types of clients (clients and big clients);

• the routing decisions involve more layers of the distribution network;

• there are inventory decisions at CD and RD.

In order to formally state the problem, let us introduce the following notation, which extends the one used to formulate WLRP, and will be used throughout the paper:

$C = \{1, \ldots, n\}$ set of the clients;
$BC = \{1, \ldots, n'\}$ set of the big clients;
$NIW = \{1, \ldots, m\}$ set of the potential transit points (the acronym stands for No Inventory Warehouse);
$IW = \{1, \ldots, m'\}$ set of the potential RD;
$W = NIW \cup IW$ set of the potential regional facilities;
$CD = \{1, \ldots, h\}$ set of the potential CD;
$P = \{1, \ldots, p\}$ set of the existing plants;
$V = \{1, \ldots, v\}$ set of the vehicles.

As before, let us introduce a directed graph $G = (N, A)$ to model the distribution network, where:
$N = BC \cup C \cup W \cup CD$ is the set of the nodes, which does not include the plants;
$N_1 = BC \cup W \cup CD$ denotes the set of nodes involved by the routing decisions of the first level;
$N_2 = C \cup W \cup CD$ denotes the set of nodes involved by the routing decisions of the second level;
A is the set of the arcs (that is the available links between nodes of the distribution network).

Furthermore, define:
D_i: demand of client i, $\forall i \in C \cup BC$;
C_{gh}: distance (in kilometers) between nodes g and $h, \forall g, h \in N$;
K_k: capacity of vehicle $k, \forall k \in V$;
WC_j: capacity of facility $j, \forall j \in W \cup CD$;
VC_j: warehousing cost per unit throughput at $j, \forall j \in W \cup CD$ (only for clients and big clients);
DC_k: transportation cost per kilometer of vehicle $k, \forall k \in V$;
TC_k: fixed cost for the usage of vehicle $k, \forall k \in V$;
FC_j: fixed cost of establishing the facility $j, \forall j \in W \cup CD$;
CP_{uj}: unit shipping cost for transferring goods from the plant u to CD $j, \forall u \in P, \forall j \in CD$;
IC_j: inventory cost for unit of good at depot j, $\forall j \in IW \cup CD$;
s_j: initial stock level at depot j, $\forall j \in IW \cup CD$;
bs_j: minimum stock level to keep at depot j (if opened), $\forall j \in IW \cup CD$.

5 The models

Starting from the general problem introduced in Section 4, here we will discuss several realistic scenarios, of increasing difficulty. In each case two mathematical programming formulations will be proposed: the first one extends the formulation of WLRP reviewed in Section 3, while the second one uses flow variables and constraints, and it is inspired by models proposed in [4].

5.1 The first scenario

Let us consider a special case of the integrated distribution network design problem, where the location decisions are limited to the layer of the network (the regional layer) where TP and RD can operate. In this scenario, only one CD is operative, which represents the only supply point of the distribution network; therefore, plants are not present.

Furthermore, we do not include inventories. Therefore, there is no distinction between RD and TP (we will refer to TP when mentioning potential facilities), and there are no minimum stock level considerations.

In this first scenario, the distribution network is thus composed by 3 layers, with routing at two levels (3/T/T). More precisely, it has the following structure:

1° layer: it contains one existing CD. We assume that there is not a capacity constraint related to the quantity of throughput at the CD. The CD supplies all the opened TP (facilities of the second layer), serves the BC and it may supply some clients (nodes of the third layer). Due to the presence of a single CD, the problem does not involve the allocation of the TP to the CD.

2° layer: it includes the potential transit points, which supply the clients.

3° layer: it is formed by clients and big clients. Since there is a single CD, the problem does not involve the allocation of the BC to the CD.

Figure 1 shows a feasible solution to this scenario.
ftbpFU2.9611in2.2243in0ptA feasible solution to the first scenarioscenario1.jpg

By the above restrictions, the first scenario specializes the integrated distribution network design problem as follows. It is the problem of determining:

• *(location decisions)* where to locate the TP, and their number;
• *(allocation decisions)* how to allocate the clients to the open facilities (CD or TP);
• *(routing decisions)* the vehicle routes for serving the clients, starting from the existing CD or from an open TP (*routing of the second level*), and the mixed tours for visiting TP and BC, starting from the existing CD (*routing of the first level*), in such a way each client and big client, as well as each open TP (in the routing of the first level), be served by exactly one vehicle, and by satisfying capacity constraints related to the vehicles;
• *(inventory decisions)* the quantity of good which must be shipped from the existing CD to the open TP, in such a way to satisfy the capacity constraints at the TP;

in order to minimize the overall costs.

Observe that the first scenario differs from WLRP essentially for the routing decisions and for the presence of a fixed cost for the usage of the vehicles (that may imply a better usage of the fleet). On the other hand, there is no limit to the maximum length of the routes. Furthermore, the first scenario addresses two types of customers: clients and big clients.

Let us propose now a three-index formulation for the first scenario by using the notation introduced in Section 4, where $W = NIW$ denotes the set of the potential transit points, and CD denotes the unique existing central depot. The formulation is based on the following set of variables, where a distinction is made between variables defining the routing of the first level, and variables defining the routing of the second level:

$$x_{ghk} = \begin{cases} 1 & if g precedes h in route of vehicle k \\ & in routing of second level, \forall g, h \in N_2, \forall k \in V \\ 0 & otherwise \end{cases}$$

$$r_{ghk} = \begin{cases} 1 & if g precedes h in route of vehicle k \\ & in routing of first level \forall g, h \in N_1, \forall k \in V \\ 0 & otherwise \end{cases}$$

$$y_{ij} = \begin{cases} 1 & if client i is allocated to facility j, \forall i \in C, \forall j \in W \cup \{CD\} \\ 0 & otherwise \end{cases}$$

$$z_j = \begin{cases} 1 & if transit point j is opened, \forall j \in W \\ 0 & otherwise \end{cases}$$

$$v_k^1 = \begin{cases} 1 & if vehicle k is used in a tour of the first level, \forall k \in V \\ 0 & otherwise \end{cases}$$

$$v_k^2 = \begin{cases} 1 & if vehicle k is used in a tour of the second level, \forall k \in V \\ 0 & otherwise \end{cases}$$

$f_{jk} \geq 0$: quantity shipped from the existing CD to the transit point j with vehicle k, $\forall j \in W, \forall k \in V$.

$$(S1) Min \sum_{j \in W} FC_j * z_j + \sum_{j \in W \cup \{CD\}} VC_j \left(\sum_{i \in C} D_i * y_{ij} \right) + \sum_{k \in V} TC_k * (v_k^1 + v_k^2) + \sum_{k \in V} \sum_{g \in N_2} \sum_{h \in N_2} C_{gh} * DC_k * x_{ghk} + \sum_{k \in V} \sum_{g \in N_1} \sum_{h \in N_1} C_{gh} * DC_k * r_{ghk} \quad (12)$$

$$\sum_{k \in V} \sum_{h \in N_2} x_{ihk} = 1 \quad \forall i \in C \quad (13)$$

$$\sum_{g \in N_2} x_{ghk} - \sum_{g \in N_2} x_{hgk} = 0 \quad \forall h \in N_2, \forall k \in V \quad (14)$$

$$\sum_{g \in S} \sum_{h \in \bar{S}} \sum_{k \in V} x_{ghk} \geq 1 \quad \forall S \subset N_2, t.c. W \cup \{CD\} \subseteq S \quad (15)$$

$$\sum_{i \in N_2} \sum_{j \in W \cup \{CD\}} x_{ijk} \leq 1 \quad \forall k \in V \quad (16)$$

$$\sum_{k\in V}\sum_{h\in N_1} r_{ihk} = 1 \quad \forall i \in BC \tag{17}$$

$$\sum_{k\in V}\sum_{h\in N_1} r_{jhk} = z_j \quad \forall j \in W \tag{18}$$

$$\sum_{g\in N_1} r_{ghk} - \sum_{g\in N_1} r_{hgk} = 0 \quad \forall h \in N_1, \forall k \in V \tag{19}$$

$$\sum_{g\in S}\sum_{h\in \bar{S}}\sum_{k\in V} r_{ghk} \geq 1 \tag{20}$$

$$\forall S \subset N_1, t.c. \{CD\} \subseteq S, \bar{S} \cap BC \neq \emptyset$$

$$\sum_{g\in S}\sum_{h\in \bar{S}}\sum_{k\in V} r_{ghk} \geq z_j \tag{21}$$

$$\forall j \in W, \forall S \subset N_1, t.c. \{CD\} \subseteq S, \bar{S} \cap \{j\} \neq \emptyset$$

$$\sum_{h\in W\cup BC} r_{CDhk} \leq 1 \;\; \forall k \in V \tag{22}$$

$$K_k \sum_{h\in N_1} r_{jhk} - f_{jk} \geq 0 \quad \forall j \in W, \forall k \in V \tag{23}$$

$$\sum_{i\in C} D_i \sum_{h\in N_2} x_{ihk} \leq K_k * v_k^2 \;\; \forall k \in V \tag{24}$$

$$\sum_{j\in W} f_{jk} + \sum_{i\in BC} D_i \sum_{h\in N_1} r_{ihk} \leq K_k * v_k^1 \;\; \forall k \in V \tag{25}$$

$$v_k^1 + v_k^2 \leq 1 \;\; \forall k \in V \tag{26}$$

$$\sum_{k\in V} f_{jk} - WC_j * z_j \leq 0 \quad \forall j \in W \tag{27}$$

$$\sum_{h\in N_2} x_{ihk} + \sum_{h\in N_2} x_{jhk} - y_{ij} \leq 1 \;\; \forall i \in C, \forall k \in V, \forall j \in W \cup \{CD\} \tag{28}$$

$$\sum_{k\in V} f_{jk} - \sum_{i\in C} D_i * y_{ij} = 0 \quad \forall j \in W \tag{29}$$

$$\begin{aligned} x_{ghk} &\in \{0,1\}, \forall g,h \in N_2, \forall k \in V \qquad (30)\\ r_{ghk} &\in \{0,1\}, \forall g,h \in N_1, \forall k \in V \\ y_{ij} &\in \{0,1\}, \forall i \in C, \forall j \in W \cup \{CD\} \\ z_j &\in \{0,1\}, \forall j \in W \\ v_k^1, v_k^2 &\in \{0,1\}, \forall k \in V \\ f_{jk} &\geq 0, \forall j \in W, \forall k \in V \end{aligned}$$

Observe that an existing transit point j can be modeled in $(S1)$ by imposing $z_j = 1$.

In order to prove the correctness of model $(S1)$, let us split its set of constraints in subsets related to:

a) the definition of the routing of the second level (constraints (13), (14) (15) (16))

b) the definition of the routing of the first level (constraints (17) (18) (19) (20) (21) (22) (23))

c) the capacity of the vehicles and the limit for their usage (constraints (24) (25) (26))

d) the capacity of the transit points (constraints (27))

e) the connection between the routing of the first and of the second level (constraints (28) (29)).

Proposition 1. Model ($S1$) is a correct formulation of the first scenario.

Proof: Let us analyze the different types of constraints, in order to prove that they correctly formulate the feasible solutions of the first scenario.

The constraints of type a) define the routing of the second level. They define tours which include all the clients (constraints (13)), and exactly one facility among the existing CD and the TP (constraints (15) and (16)). In particular, (15) are the classical subtour elimination constraints, which avoid subtours formed only by clients, without a supply node. Furthermore, by (13) each client is visited by exactly one vehicle, that is exactly one tour serves each client in C.

The constraints of type b) specify the routing of the first level. They define tours which include all the big clients (17) and the open TP (18). Each tour includes the existing CD, thanks to the subtour elimination constraints (20) and (21). Also in this case, each big client and each open TP are visited by exactly one vehicle ((17) and (18)).

It follows that, both in the routing of the first and of the second level, the tours satisfy the requirements of the first scenario. Observe that, in both cases, each vehicle may enter a node at most once, as required by the definition of tour. This is true, in particular, for the existing CD in the routing of the first level, due to constraints (22) and (19).

In this group of constraints, (23) specify that a TP j can receive a positive amount of good, f_{jk}, via a vehicle k in the routing of the first level only if k is the (unique) vehicle visiting j. In particular, by constraints (18), j must be open.

Consider now the constraints of type c). Constraints (24) and (25) are the vehicle capacity constraints. These are different for the routing of the two levels: in the routing of the first level they sum the quantities of good shipped to TP and the ones delivered to BC, while in the routing of the second level they sum the amounts of good delivered to the clients. This distinction is made possible in the model via two kinds of variables, v_k^1 and v_k^2, which specify whether a vehicle k is used for the routing of the first or of the second level, respectively. Such a distinction imposes the presence of the constraints (26), which limit the usage of each vehicle to at most one kind of routing (either of the first or of the second level) .

The constraints of type d) guarantee that each open TP receives an amount of good which does not exceed its capacity.

Finally, the constraints of type e) link the routing of the first and of the second level, via the allocation variables y_{ij}. In fact, as in the formulation WLRP, constraints (28) are used to specify what facility serves a client i in the routing of the second level: if a vehicle k visits both a client i and a facility j, then that means that client i is allocated to j (j is the only facility serving i), and y_{ij} is set to 1. Thanks to these variables, (29) specify that the amount of good received by a TP j via the routing of the first level must be equal to the total amount required by the clients allocated to j, i.e., all the clients served by tours of the second level including the facility j.

The thesis follows. ■

The problem under investigation can be alternatively formulated using flow variables and flow constraints, along the lines suggested by Hansen et al. [4].

Let us introduce the following set of variables, which model the flow of good along the distribution network, by distinguishing the flow of good in the routing of the first level from the one of the second level:

$fx_{gik} \geq 0$: quantity of good shipped through the arc (g,i) with the vehicle $k, \forall g,i \in N_2, \forall k \in V$ (routing of the second level);

$fr_{gik} \geq 0$: quantity of good shipped through the arc (g,i) with the vehicle $k, \forall g,i \in N_1, \forall k \in V$ (routing of the first level).

The flow formulation, referred to as (F1), differs from (S1) for the presence of flow constraints which replace the subtour elimination constraints. More precisely, in the routing of the second level constraints (15) are replaced by:

$$\begin{aligned} fx_{gik} &\leq x_{gik}K_k \; \forall g,i \in N_2, \forall k \in V && (31) \\ \sum_{g\in N_2} fx_{gik} - \sum_{g\in N_2} fx_{igk} &= \sum_{h\in N_2} x_{ihk} * D_i && (32) \\ \forall i &\in C, \forall k \in V && \\ \sum_{g\in N_2} fx_{gjk} - \sum_{g\in N_2} fx_{jgk} &= -\left[\sum_{i\in C}\left[\sum_{h\in N_2} x_{ihk} * D_i\right]\right] && (33) \\ \forall j &\in W \cup \{CD\}, \forall k \in V && \end{aligned}$$

Constraints (20) and (21) are replaced by:

$$fr_{gik} \leq r_{gik} * K_k \qquad \forall g,i \in N_1, \forall k \in V \tag{34}$$

$$\begin{aligned} \sum_{g\in N_1} fr_{gik} - \sum_{g\in N_1} fr_{igk} &= \sum_{h\in N_1} r_{ihk} * D_i \; \forall i \in BC, \forall k \in V && (35) \\ \sum_{g\in N_1} fr_{gjk} - \sum_{g\in N_1} fr_{jgk} &= f_{jk} \qquad \forall j \in W, \forall k \in V && (36) \end{aligned}$$

$$\sum_{g \in N_1} fr_{gCDk} - \sum_{g \in N_1} fr_{CDgk} = -\left[\sum_{j \in W} f_{jk} + \sum_{i \in BC}\left[\sum_{h \in N_1} r_{ihk} * D_i\right]\right] \forall k \in V \quad (37)$$

Due to the flow constraints, the constraints (23) of model (S1) are redundant, and they can be eliminated. Also (24) and (25) are implied by the flow capacity constraints. They can be removed from the model, together with variables v_k^1 and v_k^2 and constraints (26), provided that the following constraints are inserted in (F1):

$$\sum_{i \in N_2} \sum_{j \in W \cup \{CD\}} x_{ijk} + \sum_{i \in N_1} r_{CDik} \leq 1 \ \forall k \in V. \quad (38)$$

These constraints imply that each vehicle can enter at most one supply node and, therefore, it can be used for at most one tour (and, so, at most one kind of routing). Observe that constraints (38) imply constraints (16) and (22), which can be removed from (F1).

Furthermore, it is possible to remove the constraints of type e), i.e., (28) and (29), and therefore the allocation variables y_{ij}, by introducing an additional flow conservation constraint for each transit point, stating that the overall flow entering the TP must be equal to the overall outgoing flow:

$$\sum_{k \in V}\left(\sum_{g \in N_2} fx_{gjk} + \sum_{g \in N_1} fr_{gjk} - \sum_{g \in N_2} fx_{jgk} - \sum_{g \in N_1} fr_{jgk}\right) = 0, \forall j \in W. \quad (39)$$

Clearly, the objective function of (F1) must be rewritten as well, by substituting y_{ij} via the equations (29), and by suitably substituting the terms $(v_k^1 + v_k^2)$ by means of the design variables x_{gik} and r_{gik}.

Let us show that (F1) is a correct formulation.

Proposition 2. Model (F1) is a correct formulation of the first scenario.

Proof: The model includes the constraints of type a) of model (S1), which define the routing of the second level, with the exception of the subtour elimination constraints (15). However, the flow conservation constraints (32) and (33), together with (31), guarantee that subtours formed only by clients are not feasible, since each client requires a positive amount of flow. A source of flow, that is a facility, has therefore to be present into each feasible tour of the second level.

The same reasoning holds true for the constraints of type b), which define the routing of the first level, and which differ from the ones of model (S1) for the subtour elimination constraints. This group of constraints differs also for the removal of (23); however, these constraints are redundant in (F1), since they are implied by (34).

Consider now the constraints of type c). Constraints (24) and (25), which are not present in (F1), are implied by (31) and (34), respectively. In fact, these constraints impose that the flow carried by any vehicle along each arc, and in particular the flow outgoing each facility and the existing CD, cannot exceed the capacity of the vehicle. Moreover, the newly introduced constraints (38) guarantee that each vehicle can be used for at most one tour; therefore, they substitute (26).

The constraints of type d) are the same as in (S1).

Finally, the constraints of type e) can be correctly eliminated and substituted by the additional flow conservation constraints (39), stating that the overall flow entering an open TP via the routing of the first level must be equal to the overall flow outgoing the TP via the routing of the second level, thus linking the routing of the first and of the second level as required.

The thesis follows from the correctness of model (S1). ■

Observe that a key difference between model (F1) and model (S1) is given by the number of the constraints, which in (F1) is polynomial with respect to the problem size, and which makes (F1) tractable in practice, when solution approaches are investigated. As far as the number of the variables is concerned, (F1) uses the flow variables, which are not present in (S1); on the other hand, it does not use the variables v_k^1, v_k^2 and y_{ij}. So, depending on the number of the links, of the clients and of the potential facilities in the input distribution network, and depending on the size of the fleet of vehicles, (F1) may be also more compact than (S1) in the number of variables. A more detailed comparison of the two models will be the subject of a future work, devoted to the proposal of solution approaches to the first scenario.

5.2 The second scenario

The second scenario generalizes the first one for the number of the layers of the distribution network, which are 4 (4/R//T/T), and since the location decisions are now related to two layers of the network, and they involve CD, RD and TP. In this scenario, only one plant is assumed to be operative, which represents the only supply point of the distribution network.

Also in this case, we do not include inventories. Therefore, there is no distinction between RD and TP, and we will refer to TP when mentioning potential facilities.

In this second scenario, the distribution network has the following structure:

1° layer: it contains one existing plant. We assume that there is not a capacity constraint related to the quantity of throughput of the plant. The plant supplies all the opened CD (facilities of the second layer). Due to the presence of a single plant, the problem does not involve the allocation of the CD to the plant.

2° layer: it contains the potential CD, which operate as described in the first scenario. Now, there is the problem to allocate both the opened TP and the big clients BC to the open CD.

3° layer: it includes the potential transit points, which supply the clients as described in the first scenario.

4° layer: it is formed by clients and big clients.

Figure 2 shows a feasible solution to this scenario.
ftbpFU3.3044in2.482in0ptA feasible solution to the second scenarioscenario2.jpg

By the above discussion, the second scenario specializes the integrated distribution network design problem as follows. It is the problem of determining:

- *(location decisions)* where to locate the CD and the TP, and their number;
- *(allocation decisions)* how to allocate the big clients and the open TP to the open CD, and how to allocate the clients to the open facilities (CD or TP);
- *(routing decisions)* the vehicle routes for serving the clients, starting from an open CD or from an open TP (*routing of the second level*), and the mixed tours for visiting the open TP and the BC, starting from an open CD (*routing of the first level*), in such a way each client and big client, as well as each open TP (in the routing of the first level), be served by exactly one vehicle, and by satisfying the capacity constraints related to the vehicles;
- *(inventory decisions)* the quantity of good which must be shipped from the existing plant to the open CD, and the quantity to transfer from the open CD to the open TP, in such a way to satisfy the capacity constraints at the CD and at the TP;

in order to minimize the overall costs.

Let us extend now the three-index formulation (S1) proposed for the first scenario. Also in this case we use the notation introduced in Section 4, where $CD = \{1, \dots, h\}$ denotes the set of the potential CD, and p represents the only existing plant.

Moreover, let us introduce the following new variables:

$fp_j \geq 0$: it is the quantity of good shipped from the plant p to the CD j, $\forall j \in CD$,

$f_{ljk} \geq 0$: it is the quantity of good shipped from the CD l to the TP j with the vehicle $k, \forall l \in CD, \forall j \in W, \forall k \in V$,

and let us change the definition of the following variables:

$z_j = \begin{cases} 1 & if the facility j is opened, \forall j \in W \cup CD \\ 0 & otherwise \end{cases}$

$y_{ij} = \begin{cases} 1 & if client i is assigned to facility j, \forall i \in C \cup BC, \forall j \in W \cup CD \\ 0 & otherwise \end{cases}$

The formulation is the following:

$$(S2) \quad Min \sum_{j \in W \cup CD} FC_j * z_j + \sum_{j \in W \cup CD} VC_j \left(\sum_{i \in C \cup BC} D_i * y_{ij} \right) + \tag{40}$$

$$\sum_{j \in CD} CP_{pj} * fp_j + \sum_{k \in V} TC_k * (v_{k_1} + v_{k_2}) +$$

$$\sum_{k \in V} \sum_{g \in N_2} \sum_{h \in N_2} C_{gh} * DC_k * x_{ghk} + \sum_{k \in V} \sum_{g \in N_1} \sum_{h \in N_1} C_{gh} * DC_k * r_{ghk}$$

$$\sum_{k \in V} \sum_{h \in N_2} x_{ihk} = 1 \quad \forall i \in C \tag{41}$$

$$\sum_{g \in N_2} x_{ghk} - \sum_{g \in N_2} x_{hgk} = 0 \quad \forall h \in N_2, \forall k \in V \tag{42}$$

$$\sum_{g \in S} \sum_{h \in \bar{S}} \sum_{k \in V} x_{ghk} \geq 1 \ \forall S \subset N_2, t.c. W \cup CD \subseteq S \tag{43}$$

$$\sum_{i \in N_2} \sum_{j \in W \cup CD} x_{ijk} \leq 1 \ \forall k \in V \tag{44}$$

$$\sum_{k \in V} \sum_{h \in N_1} r_{ihk} = 1 \quad \forall i \in BC \tag{45}$$

$$\sum_{k \in V} \sum_{h \in N_1} r_{jhk} = z_j \quad \forall j \in W \tag{46}$$

$$\sum_{g \in N_1} r_{ghk} - \sum_{g \in N_1} r_{hgk} = 0 \quad \forall h \in N_1, \forall k \in V \tag{47}$$

$$\sum_{g \in S} \sum_{h \in \bar{S}} \sum_{k \in V} r_{ghk} \geq 1 \forall S \subset N_1, \tag{48}$$

$$t.c. CD \subseteq S, \bar{S} \cap BC \neq \emptyset$$

$$\sum_{g \in S} \sum_{h \in \bar{S}} \sum_{k \in V} r_{ghk} \geq z_j \ \forall j \in W, \ \forall S \subset N_1, \tag{49}$$

$$t.c. CD \subseteq S, \bar{S} \cap \{j\} \neq \emptyset$$

$$\sum_{h \in N_1} \sum_{j \in CD} r_{hjk} \leq 1 \ \forall k \in V \tag{50}$$

$$K_k * \sum_{h \in N_1} r_{jhk} - f_{ljk} \geq 0 \ \forall k \in V, \forall j \in W, \forall l \in CD \tag{51}$$

$$K_k * \sum_{h \in N_1} r_{lhk} - f_{ljk} \geq 0 \quad \forall k \in V, \forall j \in W, \forall l \in CD \tag{52}$$

$$\sum_{i \in C} D_i \sum_{h \in N_2} x_{ihk} \leq K_k * v_k^2 \ \forall k \in V \tag{53}$$

$$\sum_{l \in CD} \sum_{j \in W} f_{ljk} + \sum_{i \in BC} D_i \sum_{h \in N_1} r_{ihk} \leq K_k * v_k^1 \ \forall k \in V \tag{54}$$

$$v_k^1 + v_k^2 \leq 1 \;\; \forall k \in V \tag{55}$$

$$\begin{aligned}
\sum_{l \in CD} \sum_{k \in V} f_{ljk} - WC_j * z_j &\leq 0 \quad \forall j \in W & (56)\\
fp_j - WC_j * z_j &\leq 0 \;\; \forall j \in CD & (57)\\
\sum_{h \in N_2} x_{ihk} + \sum_{h \in N_2} x_{jhk} - y_{ij} &\leq 1 \forall i \in C, \forall k \in V, \forall j \in W \cup CD & (58)\\
\sum_{h \in N_1} r_{ihk} + \sum_{h \in N_1} r_{jhk} - y_{ij} &\leq 1 \forall i \in BC, \forall k \in V, \forall j \in CD & (59)\\
\sum_{l \in CD} \sum_{k \in V} f_{ljk} - \sum_{i \in C} D_i * y_{ij} &= 0 \quad \forall j \in W & (60)\\
\sum_{h \in W} \sum_{k \in V} f_{jhk} + \sum_{i \in C \cup BC} D_i * y_{ij} &= fp_j \quad \forall j \in CD & (61)\\
x_{ghk} &\in \{0,1\}, \forall g, h \in N_2, \forall k \in V & (62)\\
r_{ghk} &\in \{0,1\}, \forall g, h \in N_1, \forall k \in V\\
y_{ij} &\in \{0,1\}, \forall i \in C \cup BC, \forall j \in W \cup CD\\
z_j &\in \{0,1\}, \forall j \in W \cup CD\\
v_k^1, v_k^2 &\in \{0,1\}, \forall k \in V\\
f_{ljk} &\geq 0, \forall l \in CD, \forall j \in W, \forall k \in V\\
fp_j &\geq 0, \forall j \in CD
\end{aligned}$$

Proposition 3. Model (S2) is a correct formulation of the second scenario.

Proof: As in (S1), the constraints related to the routing of the second level define tours which include all the clients, and exactly one facility. Similarly, the constraints related to the routing of the first level define tours which include all the big clients and all the open TP, and exactly one CD. However, to have feasible tours, it is necessary to guarantee that the unique facility in the tours of the second level, and the unique CD in the tours of the first level, be open. This is implied by the constraints linking the routing of the second level to the one of the first level, (58) and (60), together with the capacity constraints (56), and by the constraints linking the routing of the first level to the shipment of good from the plant to the CD (61) and (59), together with the capacity constraints (57). In fact, the first kind of constraints state that the overall demand of the clients allocated to a TP j must be equal to the amount of flow received by j, through the unique tour of the first level serving j : from (56), this quantity can be positive only if j is an open TP. Similarly, the second kind of constraints state that the overall amount of good sent by a CD j to the TP and the big clients allocated to it must be equal to the quantity of good received by the plant p: from (57), this quantity can be positive only if j is an open CD. Observe that (51) and (52) guarantee that $f_{ljk} \geq 0$ only if both the CD l and the depot j are visited by the same vehicle k.

Since the remaining constraints, related to the capacity of the vehicles, are the same as in model (S1), the thesis follows. ■

Also for the second scenario, it is possible to state a flow formulation, which differs from (S2) for the presence of flow constraints that replace the subtour elimination constraints. In particular, the flow constraints are the same introduced for (F1), when substituting f_{jk} with $\sum_{l \in CD} f_{ljk}$ in constraints (36) and (37).

Also in this case, it is easy to prove that constraints (51), (53) and (54) of (S1) are redundant, and they can be eliminated. Also variables v_k^1 and v_k^2, and constraints (55), can be removed by inserting a suitable modification of (38):

$$\sum_{i \in N_2} \sum_{j \in W \cup CD} x_{ijk} + \sum_{i \in N_1} \sum_{j \in CD} r_{ijk} \leq 1 \ \forall k \in V. \tag{63}$$

Finally, it is possible to remove the constraints of type e), i.e., (58), (59), (60) and (61), and therefore the allocation variables y_{ij}, by introducing additional flow conservation constraints for each transit point and for each central depot, along the lines described for model (F1).

5.3 A special case of model (S2)

Sometimes producer companies prefer to trust some logistics activities to some specialized operators. For instance, a common usage is to commit some transportation activities, like the delivery operations to the clients. In this case, the integrated distribution network design problem does not involve routing decisions for serving the clients (that is the elements of C) but, on the other hand, it involves a (simpler) assignment problem of the clients to the facilities. The assignment costs represent now the costs of the transportation service.

The above company choice can be easily formulated by replacing the set of the constraints related to the routing of the second level, in model (S2), by a set of assignment constraints. More precisely, (41), (42), (43), (44) and (58) are replaced by: $\sum_{j \in W \cup CD} y_{ij} = 1 \qquad \forall i \in C$.

Also constraints (53) can be suitably rewritten, by using the assignment variables y_{ij}.

In the objective function, the term $\sum_{k \in V} \sum_{g \in N_2} \sum_{h \in N_2} C_{gh} * DC_k * x_{ghk}$ is replaced by $\sum_{i \in C} \sum_{j \in W \cup CD} a_{ij} * y_{ij} * D_i$. That is, the cost for serving the clients is now given in terms of the costs a_{ij}, where a_{ij} is the cost, for unit of good, payed to the operator for a delivery to client i from depot $j, \forall i \in C, \forall j \in W \cup CD$.

6 Some dynamic models

In the context of distribution network design problems, it is interesting to analyze the distribution network during a time horizon $T = \{1, 2, \ldots, \tau\}$, which is split into τ equal periods of time, rather that in a single period of time, as assumed both in the first and in the second scenario. In this case, it is significative to take into account inventory decisions.

Here we will extend the first and the second scenario over the time. In both cases, we will assume that each location decision is taken at the beginning of the time horizon T, and it can not be changed anymore.

In order to present the models, let us extend the notation introduced in Section 4 as follows:

D_{it}: is the demand of client i in period $t, \forall i \in C \cup BC, \forall t \in T$;

$\tilde{C}_t$: is the set of the clients with a positive demand in period $t, \forall t \in T$;

$\tilde{BC}_t$: is the set of the big clients with a positive demand in period $t, \forall t \in T$.

6.1 A dynamic extension of (S1)

In this extension, we admit inventories at the depots of the regional layer. Therefore, in the regional layer of the distribution network, the location decisions now include the choice of the kind of facility to open (RD or TP). At each RD, inventories are maintained in such a way to grant a certain service level; that is, we impose that the stock of good in each period of time be higher than a minimum safety stock level.

Usually, the minimum safety stock level depends on the lead time for receiving the good from a CD, and on the overall customer demand allocated to the RD. For the sake of the simplicity, we will assume here that the minimum safety stock level to maintain at depot j (if opened), $\forall j \in IW$, that is bs_j, be an input data.

Now, we can easily extend model (S1), by modifying the variables of (S1) as follows, in order to take into account the time horizon T:

$$x_{ghkt} = \begin{cases} 1 & if g precedes h in the route of vehicle k \\ & in the routing of the second level, in period t, \\ & \forall g,h \in N_2, \forall k \in V, \forall t \in T \\ 0 & otherwise \end{cases}$$

$$r_{ghkt} = \begin{cases} 1 & if g precedes h in the route of vehicle k \\ & in the routing of the first level, in period t, \\ & \forall g,h \in N_1, \forall k \in V, \forall t \in T \\ 0 & otherwise \end{cases}$$

$$y_{ijt} = \begin{cases} 1 & if client i is allocated to facility j, in period t, \\ & \forall i \in C, \forall j \in W \cup \{CD\}, \forall t \in T \\ 0 & otherwise \end{cases}$$

$$v_{kt}^1 = \begin{cases} 1 & if vehicle k is used in a tour of the first level, in period t, \\ & \forall k \in V, \forall t \in T \\ 0 & otherwise \end{cases}$$

$$v_{kt}^2 = \begin{cases} 1 & if vehicle k is used in a tour of the second level, in period t, \\ & \forall k \in V, \forall t \in T \\ 0 & otherwise \end{cases}$$

$f_{jkt} \geq 0$: quantity of good shipped from the existing CD to the depot j with vehicle k, in period $t, \forall j \in W, \forall k \in V, \forall t \in T$.

Furthermore, let us introduce the following new set of variables:

s_{jt}: stock level at depot j at the end of period $t, \forall j \in IW, \forall t \in T$.

The new model differs from (S1), first of all, in the objective function, where costs are referred to the whole time horizon (except for the fixed costs for opening facilities), and where there is a new term taking into account the inventory costs: $\sum_{j\in IW}\sum_{t\in T} IC_j * s_{jt}$.

When looking at the set of constraints, a critical point is the presence of different set of constraints for TP and for RD. Whereas the ones related to TP are similar to those in model (S1), when extended to the time horizon T, the ones related to RD must take into account the stock level, and grant that the customer service level be satisfied:

$$s_{jt-1} + \sum_{k\in V} f_{jkt} - \sum_{i\in\tilde{C}_t} D_{it} * y_{ijt} = s_{jt} \quad \forall j \in IW, \forall t \in T \tag{64}$$

$$s_{jt} - WC_j * z_j \leq 0 \quad \forall j \in IW, \forall t \in T \tag{65}$$

$$s_{jt} \geq bs_j, \forall j \in IW, \forall t \in T. \tag{66}$$

Observe that, in each period of time t, only the clients having a positive demand in t are taken into consideration. Moreover, the same client can be allocated to different facilities, in different periods of time.

Finally, we report the subtour elimination constraints, which differ from those in (S1) as follows:

$$\sum_{g\in S}\sum_{h\in\bar{S}}\sum_{k\in V} x_{ghkt} \geq 1 \tag{67}$$

$$\forall S \subset N_2, t.c. W \cup \{CD\} \subseteq S, \bar{S} \cap \tilde{C}_t \neq \emptyset, \forall t \in T$$

$$\sum_{g\in S}\sum_{h\in\bar{S}}\sum_{k\in V} r_{ghkt} \geq 1 \quad \forall S \subset N_1, \tag{68}$$

$$t.c. \{CD\} \subseteq S, \bar{S} \cap \tilde{B}C_t \neq \emptyset, \forall t \in T$$

$$\sum_{g\in S}\sum_{h\in\bar{S}}\sum_{k\in V} r_{ghkt} \geq \delta_{jt} \ \forall j \in W, \ \forall S \subset N_1, \tag{69}$$

$$t.c. \{CD\} \subseteq S, \bar{S} \cap \{j\} \neq \emptyset, \forall t \in T$$

where δ_{jt}, $\forall j \in W, \forall t \in T$, are new boolean variables taking value 1 if j serves some clients in period t. For example:

$$\sum_{i\in\tilde{C}_t} y_{ijt} \leq \left|\tilde{C}_t\right| \delta_{jt}, \forall j \in W, \forall t \in T.$$

Variables δ_{jt} can also be used to rewrite constraints (18), in order to guarantee a tour of the first level visiting a facility j in period t only if j serves some clients in period t.

6.2 A dynamic extension of (S2)

In this extension, inventories are kept both at CD and at RD. Also in this case, we assume to know the minimum safety stock level:

bs_j: minimum safety stock level to maintain at depot j (if opened) ,$\forall j \in IW \cup CD$.

We can easily extend the model (S2) by modifying the variables defined for (S2) as discussed for the previous extension, in order to take into account the time horizon T. Moreover we define:

$f_{ljkt} \geq 0$: quantity of good shipped from the CD l to the depot j with vehicle k in period $t, \forall l \in CD, \forall j \in W, \forall k \in V, \forall t \in T$;

$fp_{jt} \geq 0$: quantity of good shipped from the existing plant p to the CD j in period $t, \forall j \in CD, \forall t \in T$.

As before, we make in evidence the set of the constraints related to RD and CD, since they involve stock level considerations:

$$s_{jt-1} + \sum_{l \in CD} \sum_{k \in V} f_{ljkt} - \sum_{i \in \tilde{C}_t} D_{it} * y_{ijt} = s_{jt} \qquad \forall j \in IW, \forall t \in T \tag{70}$$

$$s_{jt-1} + fp_{jt} - \sum_{h \in W} \sum_{k \in V} f_{jhkt} - \sum_{i \in \tilde{C}_t \cup \tilde{B}C_t} D_{it} * y_{ijt} = s_{jt} \qquad \forall j \in CD, \forall t \in T \tag{71}$$

$$s_{jt} - WC_j * z_j \leq 0 \qquad \forall j \in IW \cup CD, \forall t \in T \tag{72}$$

$$s_{jt} \geq bs_j \quad \forall j \in IW \cup CD, \forall t \in T \tag{73}$$

7 Conclusions

In this work, we have studied some complex distribution network design problems, where the distribution network is made up of several layers (plants, central depots, regional depots and customers (demand points)), and which involve facility, warehousing, transportation and inventory decisions. Several realistic scenarios have been investigated. Dynamic versions have been discussed as well, where the distribution network is analyzed over a certain time horizon rather than in a fixed period of time.

For each scenario, two kinds of mathematical programming formulations have been proposed, together with a proof of their correctness. Some formulations extend models proposed by Perl and Daskin (1985) for some warehouse location-routing problems; other formulations are based on flow variables and constraints. These formulations constitute the first step towards the study of distribution network design problems which are more complex with respect to the ones studied in the literature so far. In particular, these formulations will be used as a mathematical basis to design solution methods, both exact and also approximate ones. The research is under development, and it will be the subject of a future work.

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