Università di Pisa
Dipartimento di Informatica

Technical Report: TR-01-19

# Some New Results for $k$-Dense Trees: $\mathsf{NP}$-Completeness Theory

Gianni Franceschini

14 September 2001

ADDRESS: Corso Italia 40, 56125 Pisa, Italy. TEL: +39 050 2212700 FAX: +39 050 2212726

# Some New Results for $k$-Dense Trees: NP-Completeness Theory

Gianni Franceschini

francesc@di.unipi.it

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**Abstract**

Let $k$ be a positive integer. A *$k$-cycle* is a connected graph in which each vertex has degree greater than $k$. A *$k$-dense forest* is a graph for which no subgraph is a $k$-cycle; if a $k$-dense forest is connected, then it is *$k$-dense tree*. A *$k$-leaf* is a vertex of a $k$-dense forest with degree less than or equal to $k$. Any $k$-dense forest has at least one $k$-leaf. If a $k$-leaf is removed, the resulting graph is still a $k$-dense forest. This fact is on the basis of another characterization of $k$-dense forests which make use of the concept of *$k$-elimination*, a particular ordering of removal for the vertices of a $k$-dense forest. In this paper, we consider some NP-complete problems in the area of Graph Theory. For each of these problems we study the restriction to the class of complete $k$-dense trees, for a generic integer $k$, and we try to establish whether there are some values of $k$ for which the restriction of the original problem is in P and some other values of $k$ for which this restriction still remains in the class of NP-complete problems.

**Keywords**

Graph Theory, NP-Completeness Theory, $k$-cycle, $k$-dense forest, $k$-dense tree, complete $k$-dense tree, $k$-elimination, $k$-tree, triangulated graphs, $k$-linked, Vertex Cover, Independent Set, Partition Into Triangles, Dominating Set, Graph Colorability, Disjoint Connecting Paths, Monochromatic Triangle.

## 1 Introduction

In this work we introduce "some new results for $k$-dense trees", as the title claim, pertinent to the NP-Completeness Theory. Let $k$ be a positive integer. A $k$-cycle is a connected graph in which each vertex has degree greater than $k$. A $k$-dense tree is a connected graph for which no subgraph is a $k$-cycle. In this paper we consider some NP-complete problems in the area of Graph Theory. For each of these problems we study the restriction to the class of complete $k$-dense trees, for a generic integer $k$, and we try to establish whether there are some values of $k$ for which the restriction of the original problem is in P and some other values of $k$ for which this restriction still remains in the class of NP-complete problems.

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In order to prove the NP-completeness of a problem, we will use the standard method, extensively explained in [GJ79], which make use of *polynomial transformations*. All the transformations which we will construct are from one of the following NP-complete problems.

**Problem 1.1 (3-Satisfiability or 3-SAT** [Coo71]**)**
INSTANCE*: Collection $C := \{c_1, \ldots, c_m\}$ of clauses on a finite set $U$ of Boolean variables such that $|c_i| = 3$, for all $1 \leq i \leq m$.*
QUESTION*: Is there a truth assignment for $U$ which satisfies all the clauses in $C$?*

**Problem 1.2 (Exact Cover by 3-Sets or X3C** [Kar72]**)**
INSTANCE*: A finite set $X$ with $|X| = 3t$ and a set $D \subseteq [X]^3$ of subsets of 3 elements of $X$.*
QUESTION*: Is there a set $D' \subseteq D$ such that every element of $X$ belongs to exactly one member of $D'$?*

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This paper is organized as follows. We begin (§2) with a short introduction to the Graph Theory in which we fix a set of fundamental notations and we formally define those basic concepts that we will use in the remainder of this work. In §3 we make a quick presentation of part of the basic theory concerning the class of $k$-dense trees; for those who are interested, the whole theory is contained in [Fra01a]. Subsequently we study some NP-complete problems.

**Vertex Cover & Independent Set (§4).** We prove that, for any integer $k \geq 3$, the problems *Vertex Cover* and *Independent Set* restricted to the class of complete $k$-dense trees remain NP-complete.

**Partition Into Triangles (§5).** We present an algorithm that, applied to a 2-dense forest and to a $k$-elimination for it, solve the problem *Partition Into Triangles* in linear time. Moreover we prove that, for any integer $k \geq 3$, the problem Partition Into Triangles restricted to the class of complete $k$-dense trees remains NP-complete.

**Dominating Set (§6).** We prove that, for any integer $k \geq 2$, the problem *Dominating Set* restricted to the class of complete $k$-dense trees remains NP-complete.

**Graph Colorability (§7).** We prove that, for any integer $k \geq 2$, the decision version of the problem *Graph Colorability* restricted to the class of complete $k$-dense trees is solvable in constant time.

**Disjoint Connecting Paths (§8).** We prove that, for any integer $k \geq 2$ and for some particular values of a parameter, the decision version of the problem *Disjoint Connecting Paths* restricted to the class of complete $k$-dense trees is solvable in constant time. Moreover, we introduce an algorithm that, applied to a complete $k$-dense tree and to a $k$-elimination for it, solve the general version (search) of the problem *Disjoint Connecting Paths* in linear time.

**Monochromatic Triangle (§9).** We prove that, for $k = 2$, $k = 3$ and $k = 4$, the problem *Monochromatic Triangle* restricted to the class of $k$-dense forests is solvable in constant time and we indicate a way to obtain an algorithm that, applied to a $k$-dense forest and to a $k$-elimination for it, solve the general version (search) of this restriction in linear time.

## 2 Basic Definitions and Notations

Let $I$ be a set, by $[I]^r$ we denote the set of all subsets of $I$ with cardinality equal to $r$.

An *undirected graph* (simply *graph* from now on) $G$ is a pair $(V, A)$ of sets satisfying $A \subseteq [V]^2$.

The elements of $V$ are the *vertices* (or *nodes*) of $G$, whereas the element of $A$ are the *edges* (or *arcs*).

Given a graph $G$, the set of its vertices will be denoted by $V(G)$, whereas by $E(G)$ we will denote the set of edges of $G$.

The number of the vertices of a graph is its *order*. The order of a graph $G$ will be denoted by $|G|$ and, similarly, we will indicate the number of edges of $G$ with $\|G\|$.

Let $G$ be a graph and let $a$ be an edge of $G$. Let $u$ and $v$ be two vertices $(u \neq v)$: $u$ and $v$ are the *ends* of $a$ if $a = \{u, v\}$. An edge of a graph is *incident* in a vertex if the vertex is one of its ends. Extending the notation given to denote the set of the edges of a graph $G$, we will denote the set of the edges incident in a vertex $u$, by $E(u)$.

Given a graph $G$, an edge $a$ with $x$ and $y$ as ends will be denoted by $xy$, quite apart from the kind of the graph and only in order to highlight its ends. Moreover, we could use the notation $\{x, y\}$.

Let $G := (V, A)$ and $G' := (V', A')$ be two graphs; $G$ and $G'$ are *isomorphic*, or more synthetically $G \simeq G'$, if there is a biunique function $\varphi : V \to V'$ such that:

$$xy \in A \iff \varphi(x)\varphi(y) \in A'$$

for every $x, y \in V$. Such a function is an *isomorphism* between $G$ and $G'$.

Two vertices of a graph are *adjacent* if such vertices are the ends of an edge, that is, by using the notation just introduced, if $E(u) \cap E(v) \neq \emptyset$. Two edges are *adjacent* if they share an end.

A graph $G$ is *complete* if for any two vertices $x, y$ of $G$ we have that $xy \in E(G)$. A complete graph with $r$ vertices is usually denoted by $K^r$. It can be easily derived from the definitions that a complete graph $K^r$ has $\frac{r(r-1)}{2}$ edges.

A set of vertices or edges of a graph is *independent* (o *stable*) if every pair of elements of the set is not constituted by adjacent objects.

Let $G := (V, A)$ and $G' := (V', A')$ be graphs. If $V' \subseteq V$ and $A' \subseteq A$ then $G'$ is a *subgraph* of $G$ (and $G$ is a *supergraph* di $G'$).

The relation "$G'$ is a subgraph of $G$" will be denoted by $G' \subseteq_{\mathcal{G}} G$ or it will be indicated by saying that "$G'$ is contained by $G$". If $G' \subseteq_{\mathcal{G}} G$ and $G' \neq G$ then $G'$ is a *proper* or *non-trivial* subgraph of $G$ (the same holds for $G$ as supergraph of $G'$) and in this case we will use the notation $G' \subset_{\mathcal{G}} G$. A graph $G'' := (V'', A'')$ such that $V'' \subseteq V$ and

$$A'' := \{xy \in A \mid x, y \in V''\}$$

is an *induced subgraph* of $G$ and, in particular, is *the* subgraph of $G$ induced by $V''$. In order to denote the subgraph of $G$ by a set of vertices $M$ we will write $G_M$. A graph $G''' := (V, A''')$ such that $A''' \subseteq A$ is a *partial subgraph* (or *spanning subgraph*) of $G$. Given a graph $G$, a set with $r$ vertices $D \subseteq V(G)$ which induces on $G$ a complete subgraph is an *$r$-clique*.

We will say that a graph $G$ is *minimal* (*maximal*) with some property $\mathbb{P}$, if no subgraph (supergraph) of $G$ exists for which $\mathbb{P}$ holds, by refering to the relation $\subseteq_{\mathcal{G}}$ implicitly. In a similar way, we will refer to the set inclusion relation by writing about minimal or maximal sets of vertices or edges.

Let $G := (V, A)$ be a graph. The *complement* of $G$, which is denoted by $\overline{G}$, is the graph $\left(V, [V]^2 - A\right)$.

Let $G$ be a graph, and let $u$ be a vertex of $G$. The set of the vertices of $G$ adjacent to $u$, will be denoted by $adj_G(u)$ (or, more briefly, by $adj(u)$, if $G$ is clear).

The *degree*, $deg_G(u)$ (or $deg(u)$), of a vertex $u$ is $|E(u)|$, that is the number of the edges of $G$ which have $u$ as end. If a vertex has the degree equal to zero, it is *isolated*.

A *path* is a non-empty graph $P := (V, A)$ of the form:

$$\begin{aligned} V &= \{x_0, x_1, x_2, \ldots, x_r\} \\ A &= \{x_0x_1, x_1x_2, \ldots, x_{r-1}x_r\} \end{aligned}$$

where the vertices $x_i$ are all distinct. The vertices $x_0$ and $x_r$ are *linked* (or *connected*) by $P$; the vertices $x_1, \ldots, x_{r-1}$ are the *inner vertices* of $P$. We will write about the *length* of a path $P$, that is the number of the edges of $P$, by refering to $\|P\|$. We will often identify a path with one of the two natural sequences of its vertices, for instance $P := x_0x_1x_2 \ldots x_r$.

In order to reference subpaths of a path, we will write (for a path $P$ and for $0 \leq i \leq j \leq r$):

$$Px_j := x_0 \dots x_j$$
$$x_iP := x_i \dots x_r$$
$$x_iPx_j := x_i \dots x_j$$

To denote the concatenation of paths, we will write, say, $PxQyR$ instead of $Px \cup xQy \cup yR$. Given a graph $G$, a path $P := x_0 \dots x_r$ in $G$ and a vertex $f$ such that $f \in V(G)$, $f \notin P$ and there is $fx_0 \in E(G)$ ($fx_r \in E(G)$), we will denote the path:

$$P' := fx_0 \dots x_r \quad (P' := x_0 \dots x_r f)$$

by $fP$ ($Pf$).

Two or more paths are *independent* if none of them contains an inner vertex of another. Given a graph $H$, a path $P := x_0 \dots x_r$ which shares with $H$ only $x_0$ and $x_r$, is an *$H$-path*.

A graph $G$ is *connected* if for any $\{u, v\} \in [V(G)]^2$ there is a path $P \subseteq_{\mathcal{G}} G$ which link $u$ and $v$. The *connected components* of a graph are its maximal connected subgraphs.

Let $P := (V, A) = x_0x_1 \dots x_r$ be a path with $r \geq 2$, the graph:

$$C := (V, A \cup \{x_rx_0\})$$

is a *cycle* and we will often denote a cycle with a cyclic sequence of its vertices, for instance $C := x_0x_1x_2 \dots x_rx_0$.

There is another characterization of cycles which has relevance to the concept of $k$-cycle: a graph $C$ is a *cycle* if and only if the following conditions are valid:

- $C$ is connected,
- $deg_C(v) = 2$, for every $v \in V(C)$.

The *length* of a cycle $C$ is the number of its edges (or its vertices, since $\|C\| = |C|$). A graph $G$ is *acyclic* if there is not any cycle $C$ such that $C \subseteq_{\mathcal{G}} G$.

A *forest* is an acyclic graph. A connected forest is a *tree*

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Let $G$ be a graph, the greatest integer $r$ such that $K^r \subseteq_{\mathcal{G}} G$ is the *clique number* of $G$ and it will be denoted by $\omega(G)$.

An *$r$-colouring* of the vertices of a graph $G := (V, A)$ is a function $c : V \to \{1, \dots, r\}$ such that $c(v) \neq c(u)$ if $v$ and $u$ are adjacent, for each $v, u \in V$. The elements of set $\{1, \dots, r\}$ are the available *colours*. A graph for which there is a $r$-colouring is *$r$-colourable*. The smallest integer $r$ for which $G$ is $r$-colourable is its *chromatic number* and it will be denoted by $\chi(G)$.

# 3 Dense Forests & Dense Trees

**Definition 3.1**
*A connected graph $C := (V_C, A_C)$ is a $k$-*cycle*, with $k$ integer $1 \leq k < |V_C| - 1$, if, for each $v \in V_C$, the following condition holds:*

$$deg_C(v) > k.$$

**Observation 3.2**
*For all integer $k \geq 2$, a $k$-*cycle* is also a $r$-*cycle* with $1 \leq r < k$.*

**Definition 3.3**
*A $k$-*dense forest* $F$, with $k \geq 1$ integer, is a graph without $k$-cycles.*

**Definition 3.4**
*A $k$-*dense tree* $T$, with $k \geq 1$ integer, is a connected graph without $k$-cycles.*

Moreover, any connected graph of order $n \leq k$ satisfies the conditions of the definition of $k$-dense tree, in fact it is obvious that a graph with these characteristics cannot contain a $k$-cycle. Dense trees of this kind are *trivial*.

We can derive the following, simple observation.

**Observation 3.5**
*The connected components of a $k$-dense forest are $k$-dense trees.*

Consequently we will call *trivial* those $k$-dense forests which have only trivial $k$-dense trees as connected components.

By Observation 3.2 we have:

**Observation 3.6**
*A $k$-*dense forest *is also a $r$-*dense forest*, with $r > k$.*

And similarly:

**Observation 3.7**
*A $k$-*dense tree *is also a $r$-*dense tree*, with $r > k$.*

**Proposition 3.8**
*A $k$-*dense tree *$T$ must have a vertex $f$ such that $deg_T(f) \leq k$.*

PROOF. By contradiction. Suppose that all the vertices $v \in V(T)$ have $deg_T(v) > k$. Then $T$, which is connected, is a $k$-cycle, against the hypothesis. □

**Definition 3.9**
*A $k$-*leaf *of a $k$-dense forest $F$ is a vertex $f \in V(F)$ such that $deg_F(f) \leq k$. If $deg_F(f) = k$, the $k$-leaf is* complete*. The set of the $k$-leaves of $F$ is denoted by $\mathcal{L}_F$.*

**Lemma 3.10**
*Let $F := (V, A)$ be a $k$-dense forest. Any subgraph of $F$ is a $k$-dense forest.*

PROOF. Immediate: if a subgraph of $F$ contained a $k$-cycle, then $F$ would also contain it, against the hypotheses. □

The following result is only a particular case of the previous one. We introduce it so as to mantain the uniformity with the subsequent theory.

**Lemma 3.11**
*Let $F := (V, A)$ be a $k$-dense forest and let $f$ be a $k$-leaf of $F$. The subgraph $F_{V-\{f\}}$ of $F$ induced by $V - \{f\}$ still is a $k$-dense forest.*

PROOF. It follows directly from Lemma 3.10. □

In general, a property analogous to the above one, does not hold for $k$-dense trees. We will see that a property analogous to the one of Lemma 3.11 holds for an interesting subclass of $k$-dense trees.

**Definition 3.12**
*Let $G := (V, A)$ be a graph of order $n$. A $k$-*elimination *$\sigma := [v_1, \ldots, v_n]$, with $k \geq 1$ integer, is an ordering of the vertices of $G$ such that, for all integer $i$ with $1 \leq i \leq n$,*

$$deg_{G_{V_i}}(v_i) \leq k$$

*where $G_{V_i}$ is the subgraph of $G$ induced by $V_i := \{v_j \mid i \leq j\}$.*

**Notation 3.13**
*Let $G := (V, A)$ be a graph with $n$ vertices and let $\sigma := [v_1, v_2, \ldots, v_n]$ be a $k$-elimination for $G$, we introduce the following notations:*

- *by $\sigma(i)$ we denote the vertex of $G$ in the $i$-th position in $\sigma$ (from $v_1$);*
- *given a set of vertices $U \subseteq V$, by $\sigma_U$ we denote the sub-sequence of $\sigma$ induced by the vertices in $U$;*
- *given a vertex $v \in V$, by $\langle\sigma\rangle^{-1}(v)$ we denote the position (the index) of $v$ in $\sigma$;*
- *given an integer $i$, with $1 \leq i \leq n$, by $G_i$ we denote the subgraph of $G$ induced by $\{v_j \mid i \leq j \leq n\}$;*
- *given a vertex $v \in V$, we set $\Lambda^\sigma(v) := \Lambda^\sigma_G(v) := \left\{u \in adj_G(v) \;\middle|\; \langle\sigma\rangle^{-1}(u) > \langle\sigma\rangle^{-1}(v)\right\}$.*

*In order to denote some particular $k$-eliminations, we use some "list-like" notations. Given $\sigma := [v_1, \ldots, v_n]$, $\sigma' := [u_1, \ldots, u_m]$ and a vertex $x$ (or, more generally, $t$ vertices $x_1, \ldots, x_t$) not contained in $\sigma$, by:*

$$[x \mid \sigma] \qquad ([x_1, \ldots, x_t \mid \sigma])$$

*we denote:*

$$[x, v_1, \ldots, v_n] \qquad ([x_1, \ldots, x_t, v_1, \ldots, v_n])$$

*whereas by app $(\sigma, \sigma')$ we denote:*

$$[v_1, \ldots, v_n, u_1, \ldots, u_m].$$

*Finally, we denote $[v_2, \ldots, v_n]$ by tail $(\sigma)$.*

**Definition 3.14**
*A $k$-*dense forest *$F$, with $k \geq 1$ integer, is a graph for which a $k$-elimination exists.*

**Theorem 3.15 (**[Fra01a]**)**
*A graph is a $k$-dense forest according to Definition 3.3 if and only if it is a $k$-dense forest according to Definition 3.14.*

**Observation 3.16**
*Let $F$ be a $k$-dense forest and let $\sigma$ be a $k$-elimination for $F$. Let $F'$ be a subgraph of $F$, by Lemma 3.10, we know that $F'$ is a $k$-dense forest. The ordering $\sigma_{V(F')}$ is a $k$-elimination for $F'$.*

**Proposition 3.17 (**[Fra01a]**)**
*Let $T$ be a $k$-dense tree of order $n$, the following condition holds:*

$$n - 1 \leq \|T\| \leq M \quad \textit{where} \quad M := \begin{cases} nk - \frac{k(k+1)}{2} & n \geq k+1 \\ \frac{n(n-1)}{2} & n < k+1 \end{cases}$$

*and, both for the lower and upper bound, there are cases in which the equality holds. Moreover, the upper bound holds for $k$-dense forests also.*

**Proposition 3.18**
*Let $F$ be a $k$-dense forest, then $\chi(F) \leq k+1$.*

Proof. See [Fra01a]. □

**Definition 3.19**
*A* complete $k$-dense tree *is a $k$-dense tree with the maximum number of edges (see Proposition 3.17).*

A property, analogous to the one of Lemma 3.11, holds for the class of complete $k$-dense trees. We have:

**Lemma 3.20 (**[Fra01a] or [Fra01b]**)**
*Let $T := (V, A)$ be a complete $k$-dense tree and let $f$ be a $k$-leaf of $T$. The subgraph $T_{V-\{f\}}$ of $T$ induced by $V - \{f\}$ still is a complete $k$-dense tree.*

**Definition 3.21**
*A* complete $k$-dense tree *$T$, with $k \geq 1$ integer, is a graph for which there is a $k$-elimination $\sigma$ such that, given $n := |T|$:*

*(i)* $\forall i \,.\, 1 \leq i \leq n-k \Rightarrow deg_{T_i}(\sigma(i)) = k$,

*(ii)* $\forall i \,.\, max\,\{n-k+1, 1\} \leq i \leq n-1 \Rightarrow deg_{T_i}(\sigma(i)) = n-i$.

*Condition (ii) can be replaced with: $T_{max\,\{n-k+1,1\}} \simeq K^{min\,\{n,k\}}$.*

**Theorem 3.22 (**[Fra01a]**)**
*A graph is a complete $k$-dense tree according to Definition 3.19 if and only if it is a complete $k$-dense tree according to Definition 3.21.*

**Proposition 3.23 (**[Fra01a]**)**
*Any $k$-elimination for a complete $k$-dense tree has the form stated Definition 3.21.*

**Proposition 3.24 (**[Fra01a] or [Fra01b]**)**
*Let $T$ be a non-trivial complete $k$-dense tree, then $\omega(T) = k + 1$.*

**Proposition 3.25**
*Let $T$ be a non-trivial complete $k$-dense tree, then $\chi(T) = k + 1$.*

**Definition 3.26**
*A $k$-tree $T$, with $k \geq 1$ integer, is a graph for which there is a $k$-elimination $\sigma$ such that, given $n := |T|$:*

*(i)* $\forall i \,.\, 1 \leq i \leq n - k \Rightarrow deg_{T_i}(\sigma(i)) = k$,

*(ii)* $\forall i \,.\, max\,\{n - k + 1, 1\} \leq i \leq n - 1 \Rightarrow deg_{T_i}(\sigma(i)) = n - i$,

*(iii)* $\forall i \,.\, 1 \leq i \leq n - k \Rightarrow adj_{T_i}(\sigma(i))$ *is a clique.*

*The second condition can be replaced with:* $T_{max\{n-k+1,1\}} \simeq K^{min\{n,k\}}$.

A graph $G$ of order $n$ is *triangulated* if none of its induced subgraphs is a cycle of length $n > 3$. An ordering $\epsilon := [v_1, \ldots, v_n]$ of the vertices of $G$ is a *perfect elimination* if for each $v_i$ the set:

$$\{v_j \in adj_G(v_i) \mid j > i\}$$

is a clique. So, by Definition 3.26, the $k$-trees are triangulated.

**Proposition 3.27 (**[Fra01a]**)**
*Any $k$-elimination for a $k$-tree is of the kind required in Definition 3.26.*

**Observation 3.28**
*A member of any of the classes of $k$-dense forests, $k$-dense trees, complete $k$-dense trees and $k$-trees can be recognized in $O(n \log n)$ time (of course with some different constant factor in $k$ which depends on the particular class) that is, for any $k$-dense forest, complete $k$-dense tree or $k$-tree it is possible to find a $k$-elimination which have the properties specific for the particular class (see [Fra01a]).*

# 4 Vertex Cover & Independent Set

**Problem 4.1 (Vertex Cover or VC** [Kar72]**)**
INSTANCE*: A graph $G$ and a positive integer $H \leq |G|$.*
QUESTION*: Is there a* vertex cover *for $G$ with cardinality less or equal to $H$? That is, is there a set $V' \subseteq V(G)$ with $|V'| \leq H$ such that, for each edge $\{u, v\} \in E(G)$, the condition $\{u, v\} \cap V' \neq \emptyset$ holds?*

We start considering a classical NP-complete problem, the Vertex Cover (VC for short), we have:

**Theorem 4.2**
*For $k \geq 3$, the problem Vertex Cover restricted to complete $k$-dense trees still is* NP*-complete.*

Proof. Since VC is a NP-complete problem, its version restricted to complete $k$-dense trees belongs to NP.

We will define a transformation from 3-SAT to the problem VC restricted to complete $k$-dense trees. Given a set:

$$U := \{u_1, \ldots, u_n\}$$

of Boolean variables and given a set:

$$C := \{c_1, \ldots, c_m\}$$

of clauses such that $c_i \subseteq U$ and $|c_i| = 3$ for all $1 \le i \le m$, we will construct a complete $k$-dense tree $T$ and we will define a positive integer $H \le |T|$ such that:

> "there is a truth assignment for $U$ which satisfies all the clauses in $C$"
>
> $$\Longleftrightarrow$$
>
> "there is a set $V' \subseteq V(T)$ with $|V'| \le H$ such that, for each edge $\{u, v\} \in E(T)$, the condition $\{u, v\} \cap V' \neq \emptyset$ holds."

Let $Q$ be a complete graph of order $k$ with:

$$V(Q) := \{q_1, \ldots, q_k\}.$$

Let us define the set of $2n$ vertices:

$$L := \{l \mid l \in \{u_1, \ldots, u_n\} \cup \{\overline{u}_1, \ldots, \overline{u}_n\}\}.$$

For each vertex $u_i \in L$, let us define the following set of $k$ edges:

$$A_{u_i} := \{u_i q \mid q \in V(Q)\}.$$

For each vertex $\overline{u}_i \in L$, let us define the following set of $k$ edges:

$$A_{\overline{u}_i} := \{\overline{u}_i u_i\} \cup \{\overline{u}_i q \mid q \in (V(Q) - \{q_1\})\}.$$

For each 3-clauses $c_j = \{l_{j,1}, l_{j,2}, l_{j,3}\}$, let us define the following set of vertices:

$$V_{c_j} := \{s_{j,1}, s_{j,2}, s_{j,3}\}$$

and the following sets of $k$ edges, which are associated to the three vertices of $V_{c_j}$:

$$A_{s_{j,1}} := \{s_{j,1} l_{j,1}\} \cup \{s_{j,1} q \mid q \in (V(Q) - \{q_1\})\}$$

$$A_{s_{j,2}} := \{s_{j,2} s_{j,1}\} \cup \{s_{j,2} l_{j,2}\} \cup \{s_{j,2} q \mid q \in (V(Q) - \{q_1, q_2\})\}$$

$$A_{s_{j,3}} := \{s_{j,3} s_{j,1}, s_{j,3} s_{j,2}\} \cup \{s_{j,3} l_{j,3}\} \cup \{s_{j,3} q \mid q \in (V(Q) - \{q_1, q_2, q_3\})\}$$

Obviously, in the above defined 3 sets of edges, each occurrence $l_{j,i}$ refers to the vertex in $L$ which have as label the literal $l_{j,i}$ of the clause $c_j$.

Finally, let us consider the set $B := \{b_1, b_2\}$ ad the set of $2k$ edges defined as follows:

$$A_B := \bigcup_{b \in B} \{bq \mid q \in V(Q)\}.$$

The graph $T$ such that:

$$\begin{aligned} V(T) &= V(Q) \cup B \cup L \cup \left( \bigcup_{j=1\ldots m} V_{c_j} \right) \\ E(T) &= E(Q) \cup A_B \cup \left( \bigcup_{i=1\ldots n} A_{u_i} \cup A_{\overline{u}_i} \right) \cup \left( \bigcup_{j=1\ldots m} (A_{s_{j,1}} \cup A_{s_{j,2}} \cup A_{s_{j,3}}) \right) \end{aligned}$$

is a complete $k$-dense tree. In fact, if we eliminate the vertices of $T$ in order inverted with respect to the one of the definition, that is if we consider the following ordering of the vertices:

$$\begin{aligned} \sigma := \quad & [b_2, b_1, \\ & s_{m,3}, s_{m,2}, s_{m,1}, \\ & \ldots \\ & s_{1,3}, s_{1,2}, s_{1,1}, \\ & \overline{u}_n, \ldots, \overline{u}_1, \\ & u_n, \ldots, u_1, \\ & q_k, \ldots, q_1] \end{aligned}$$

we obtain a $k$-elimination with the properties required by Definition 3.21.

To complete the transformation, let us set:

$$H := k + n + 2m.$$

The complete $k$-dense tree $T$ has $t := k + 2 + 2n + 3m$ vertices and, by Proposition 3.17, $tk - \frac{k(k+1)}{2}$ edges. Therefore $T$ can be constructed in polynomial time in the parameters $n$ and $m$ of any instance of the problem 3-SAT.

Obviously the graph $T$ is not a $k$-tree; otherwise the thesis would certainly be false. In fact, (for instance see [Gol80] pp. 98-100) an algorithm is known that, for any triangulated graph, finds an independent set of vertices of maximum cardinality in polynomial time; by Lemma 4.4, such an algorithm could be used to find in polynomial time a vertex cover of minimum cardinality for each triangulated graph and then for each $k$-tree.

In order to prove that $T$ is not a $k$-tree, let us consider a clause $c_j = \{l_{j,1}, l_{j,2}, l_{j,3}\}$ and the vertices of $T$ associated to it. From the definition of $T$ it follows that:

$$\{l_{j,2}, s_{j,1}\} \subset \Lambda^{\sigma}(s_{j,2})$$

since $s_{j,1}$ is adjacent only to a vertex $l \in L$ and since $l \neq l_{j,2}$ ($|c_j| = 3$), the vertices $l_{j,2}$ and $s_{j,1}$ are not adjacent; then $\Lambda^{\sigma}(s_{j,2})$ is not a $k$-clique. Then the $k$-elimination $\sigma$ for $T$ does not have the properties required by Definition 3.26 and, by Proposition 3.27, we can conclude that $T$ is not a $k$-tree.

Now, we have to prove the double statement that we have initially introduced.

($\Leftarrow$) Let us suppose that there is a set $V' \subseteq V(T)$ with $|V'| \leq H$, and such that, for each edge $\{u, v\} \in E(T)$, the following holds

$$\{u, v\} \cap V' \neq \emptyset.$$

Let us prove three fundamental fact at once.

(a) For each set $U_i := \{u_i, \overline{u}_i\}$ we are sure that:

$$U_i \cap V' \neq \emptyset.$$

In fact, since $u_i\overline{u}_i \in E(T)$, if no vertex of $U_i$ were contained in $V'$, such set of vertices would not be a vertex cover for $T$, against the hypothesis.

(b) For each set $V_{c_j} = \{s_{j,1}, s_{j,2}, s_{j,3}\}$ we know that:

$$\left|V_{c_j} \cap V'\right| \geq 2.$$

In fact $V_{c_j}$ is a 3-clique of $T$, then at least two vertices of $V_{c_j}$ must be contained in $V'$ so that all the three edges connecting the vertices in $V_{C_j}$ have at least one end contained in $V'$.

(c) Finally, we know that:

$$V(Q) \subseteq V'.$$

Let us suppose that this is not true. Since $V(Q)$ is $k$-clique of $T$, the following must hold:

$$|V(Q) \cap V'| \geq k - 1$$

otherwise $V'$ would not be a vertex cover for $T$. Let $q_i$ be the vertex of $Q$ not included in $V'$; since $V'$ is a vertex cover for $T$, it must contain the two vertices of $B$, otherwise the two edges $b_1q_i$ and $b_2q_i$ would not have any end in $V'$, since $q_i \notin V'$. However, as seen before, at least $n$ vertices of $L$ and at least $2m$ vertices of $\bigcup_{j=1\ldots m} V_{c_j}$ are contained in $V'$, together with the two vertices of $B$ and with the $k-1$ vertices of $V(Q)$. Then, we have:

$$|V'| \geq k + 1 + n + 2m > H$$

against the hypothesis about the cardinality of $V'$.

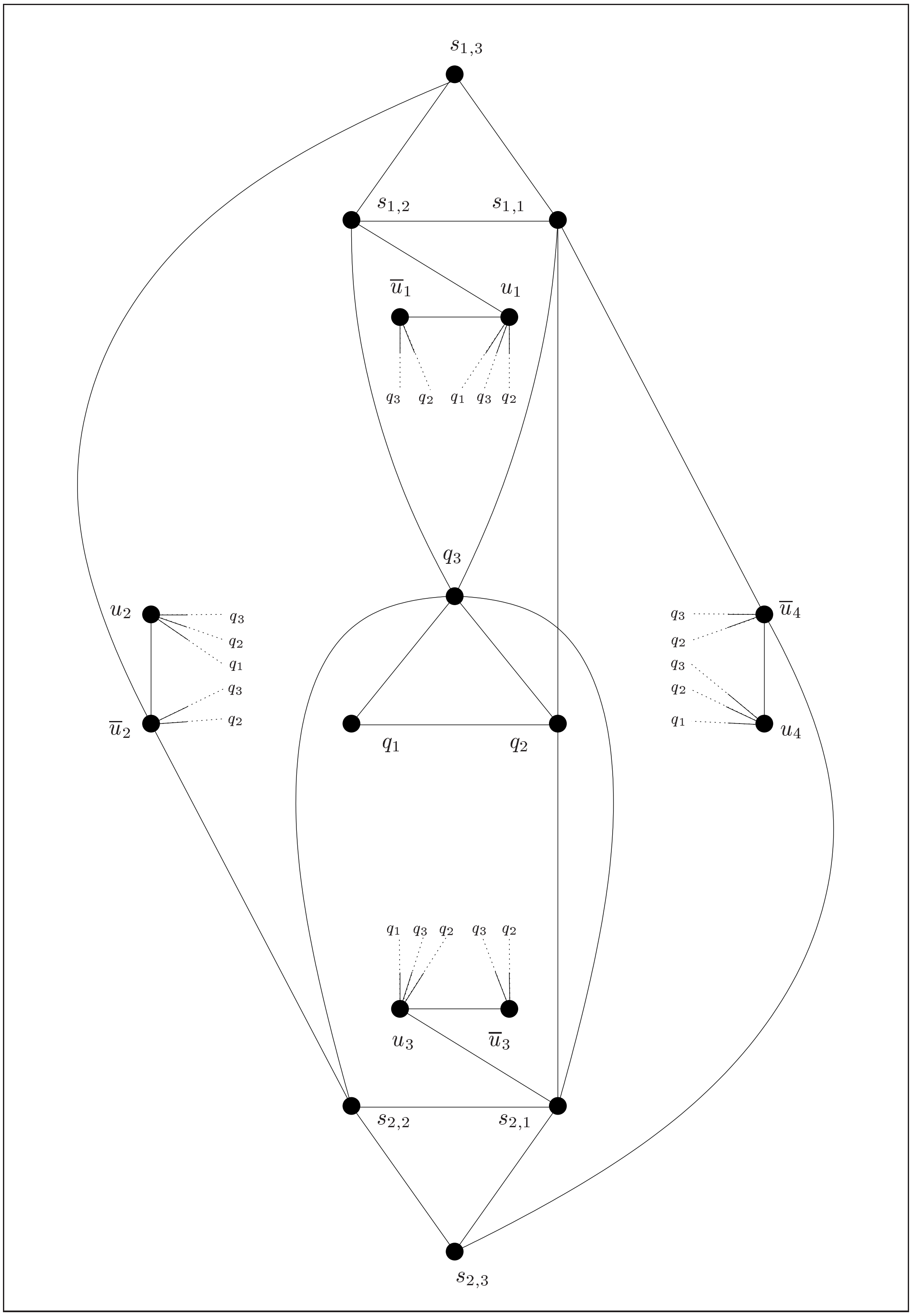


**Figure 4.1:** The complete $k$-dense tree $T$ associated to the instance of 3-SAT defined by $U := \{u_1, u_2, u_3, u_4\}$ and $C := \{\{\overline{u}_4, u_1, \overline{u}_2\}, \{u_3, \overline{u}_2, \overline{u}_4\}\}$ (we omit the vertices in $B$).

Therefore, by (a), (b) and (c), we have:

$$|V'| \geq k + n + 2m = H$$

whereas, for the initial hypothesis, we have, $|V'| \leq H$. leading to the equality $|V'| = H$. Therefore:

(i) for each set $\{u_i, \overline{u}_i\}$, only one of its vertices belongs to $V'$,

(ii) for each set $V_{c_j}$, only two of its vertices belongs to $V'$.

Consequently the assignment function $\mathcal{S} : U \to \{True, False\}$ such that, for each $u_i \in U$, the following holds:

$$\mathcal{S}(u_i) := \begin{cases} True & u_i \in V' \\ \\ False & \overline{u}_i \in V' \end{cases}$$

is well-defined.

Now, we have to prove that all the clauses in $C$ are satisfied by $\mathcal{S}$. Let $c_j \in C$. By the hypothesis on $V'$, we know that, for each edge $sl$ with $s \in V_{c_j}$ and $l \in c_j$, at least one of its ends belongs to $V'$. By (ii), we know that only two vertices of $V_{c_j}$ are contained in $V'$; then, if we denote the vertex of $V_{c_j}$, not included in $V'$, by $s'$, the edge $s'l'$, with $l' \in c_j$, must have the end $l'$ in $V'$. Therefore, by the definition of the function $\mathcal{S}$, the literal $l'$ is satisfied together with the clause $c_j$ containing it.

—————

($\Rightarrow$) Let $\mathcal{S}_U : U \to \{True, False\}$ be an assignment for the Boolean variables in $U$ which satisfies every clause in $C$. Let us define the following set of $n$ vertices of $T$:

$$V_U := \{l \mid l \in L \wedge \text{“\textit{the literal $l$ is satisfied by $\mathcal{S}_U$}”}\}.$$

For each clause $c_j \in C$, let us choose a vertex $s_j \in V_{c_j}$ such that:

$$adj_T(s_j) \cap V_U \neq \emptyset$$

(obviously, a vertex with these characteristics exists, since $\mathcal{S}_U$ satisfies all the clause of $C$) and let us define the set of $2m$ vertices:

$$V_C := \bigcup_{t=1\ldots m} V_{c_t} - \{s_t\}.$$

The set of $k + n + 2m$ vertices:

$$V'' := V(Q) \cup V_U \cup V_C$$

is a vertex cover for $T$. Let us verify this claim for each of the sets of edges that we have previously defined.

- Each edge in $E(Q)$ or in $A_B$ has one end in $V''$, since $V(Q) \subset V''$.
- For each vertex $u_i$, all the edges in $A_{u_i}$ are of the kind $u_i q$ with $q \in V(Q)$ and therefore, since $V(Q) \subset V''$, they have at least one end in $V''$.
- For each vertex $\overline{u}_i$, we have that:

  $$A_{\overline{u}_i} = \{\overline{u}_i u_i\} \cup \{\overline{u}_i q \mid q \in (V(Q) - \{q_1\})\}.$$

  Since all the vertices of $L$ whose corresponding literals are satisfied by $\mathcal{S}_U$, are included in $V_U$, one of the ends of the edge $\overline{u}_i u_i \in A_{\overline{u}_i}$ is certainly contained in $V_U$, hence in $V''$. The same as the previous case holds for the other edges in $A_{\overline{u}_i}$.
- For each set $V_{c_j} = \{s_{j,1}, s_{j,2}, s_{j,3}\}$, we know that two of its vertices are contained in $V_C$ and so all the edges in:

$$A_{c_j} := A_{s_{j,1}} \cup A_{s_{j,2}} \cup A_{s_{j,3}}$$

with both the ends in $V_{c_j}$ are correctly covered by $V''$. The same holds for every edge in $A_{c_j}$ having an end in $V(Q)$, since $V(Q) \subset V''$.

Now, we have to examine the three edges of $A_{c_j}$ having one end in $V_{c_j}$ and the other one in $L$. Since two vertices of $V_{c_j}$ are included in $V_C$, two of this edges have one end in $V''$. Moreover, for the definition of $V_C$, we know that the following holds for the vertex $s_j \in V_{c_j}$ which is not included in $V_C$:

$$adj_T(s_j) \cap V_U \neq \emptyset$$

and so the remaining edge among those with the ends in $V_{c_j}$ and in $L$, has also the end in $L$ included in $V_U$ (and so in $V''$). □

**Problem 4.3 (Independent Set or IS)**
INSTANCE: *A graph $G$ and a positive integer $H \leq |G|$.*
QUESTION: *Is there a subset of the vertices of $G$ which is an independent set of cardinality greater or equal to $H$? That is, is there a set $V' \subseteq V(G)$ with $|V'| \geq H$ such that for each $v_1, v_2 \in V'$, the edge $v_1 v_2$ is not included in $E(G)$?*

**Lemma 4.4 ([GJ79])**
*Let $G := (V, A)$ be a graph and let $V' \subseteq V$. The following claims are equivalent:*

1. *$V'$ is a vertex cover for $G$.*
2. *$V - V'$ is an independent set of $G$.*
3. *$V - V'$ is a clique for $\overline{G}$, the complement graph of $G$.*

**Theorem 4.5**
*For $k \geq 3$, the problem Independent Set restricted to complete $k$-dense trees still is NP-complete.*

PROOF. It directly follows from Lemma 4.4 and from Theorem 4.2. □

## 5 Partition Into Triangles

**Problem 5.1 (Partition Into Triangles or PIT)**
INSTANCE: *A graph $G$ with $|G| = 3b$, with $b$ positive integer.*
QUESTION: *Can the set $V(G)$ be partitioned into $b$ disjoint sets $V_1, \ldots, V_b$ such that, for each $V_i$, $G_{V_i} \simeq K^3$?*

**Theorem 5.2**
*The problem PIT restricted to 2-dense forests is solved by Algorithm 5.1.*

PROOF. Let $F$ be a 2-dense forest such that $|F| = 3b$ with $b$ positive integer, and let $\sigma$ be a 2-elimination for $F$. We will prove the thesis by induction on $|F|$.

If $|F| = 3$ then $F$ is partitionable in triangles if and only if it is a complete graph. Therefore, the Algorithm 5.1 solves the problem correctly by conditional steps 1 and 2.

If $|F| = n > 3$, let us suppose that, for any 2-dense forest of order $m < n$, the Algorithm 5.1 solves the problem PIT correctly. By Definition 3.12, we know that the vertex $v$ selected in step 8 is a 2-leaf of $F$ and so we have that $deg_F(v) \leq 2$. The 2-dense forest $F$ is partitionable in triangle if and only if the following coditions hold:

(i) the set $\Lambda^\sigma(v)$ is a 2-clique,

(ii) the graph $F_{V(F)-(\{v\}\cup\Lambda^\sigma(v))}$ is partitionable in triangles.

Obviously, the condition (i) is verified correctly by conditional steps 9 and 13 of the Algorithm 5.1.

By Lemma 3.10, we know that the graph $F'$ defined in step 16 is a $k$-dense forest of order $n-3$. Moreover, by Observation 3.16, we know that the ordering $\sigma'$ defined in step 17

**Algorithm 5.1** $\mathsf{PIT}_2$

**Input:** $\Rightarrow$ $F$, 2-dense forest such that $|F| = 3b$ with $b$ positive integer,

$\Rightarrow$ $\sigma$, 2-elimination for $F$.

**Output:** $\Leftarrow$ if $F$ is partitionable in triangles then the algorithm returns a collection of sets of three vertices, otherwise the algorithm fails.

```
if |F| = 3 then
  if F ≃ K^3 then
    return {V(F)}
  else
    fail
  end if
else
  v ← σ(1)
  if |Λ^σ(v)| ≤ 1 then
    fail
  else
    ⟨Let v_1 and v_2 be the vertices in Λ^σ(v)⟩
    if v_1 v_2 ∉ E(F) then
      fail
    else
      F' ← F_{V(F)−{v,v_1,v_2}}
      σ' ← σ_{V(F)−{v,v_1,v_2}}
      P' ← PIT_2(F', σ')
      return P' ∪ {{v, v_1, v_2}}
    end if
  end if
end if
```

is a $k$-elimination for $F'$. Therefore, by the induction hypothesis, the Algorithm 5.1 solves the problem PIT correctly for the 2-dense forest $F'$ hence also the condition (ii) is verified correctly. □

**Theorem 5.3**
*Let $F$ be a 2-dense forest and $\sigma$ be a 2-elimination for $F$. The Algorithm 5.1 applied to $F$ and $\sigma$ solves the problem PIT in $O(|T|)$ time.*

PROOF. Trivial. □

**Theorem 5.4**
*For $k \geq 3$, the problem Partition Into Triangles restricted to complete $k$-dense trees remains* NP*-complete.*

PROOF. Since PIT is an NP-complete problem, also its version restricted to complete $k$-dense trees belongs to NP.

We will define a transformation from X3C to the problem PIT restricted to complete $k$-dense trees. Given a finite set $X$ with $|X| = 3t$ and a set $D \subseteq [X]^3$ of subsets of three elements of $X$, we will define a complete $k$-dense tree $T$ of order $3b$, with $b$ positive integer, such that:

"there is a set $D' \subseteq D$ such that every element of $X$ belongs to exactly one member of $D'$"

$$\Longleftrightarrow$$

"the set $V(T)$ can be partitioned into $b$ disjoint sets $V_1, \ldots, V_b$ such that, for each $V_i$, $T_{V_i} \simeq K^3$."

Let $Q$ be a complete graph of order $k$ with $V(Q) := \{q_1, \ldots, q_k\}$. Let $s$ and $r$ be the unique natural numbers such that:

$$k = 3s + r$$

with $r < 3$. If $r > 0$, that is if $k$ is not divisible by 3, obviously the vertices of the graph $Q$ are not partitionable in triangles. In order to eliminate this possibility, let us define a set:

$$S := \{q_{k+1}, \ldots, q_{k+d}\}$$

of $d := 3 - r$ vertices and, for each $q_i \in S$, a set of $k$ edges:

$$A_{q_i} := \{q_i q_j \mid i - k \leq j < i\}.$$

Let $P_1, \ldots, P_5$ be five sets of $3k$ vertices; each set $P_i$ consists of $k$ groups of 3 vertices:

$$H_{i,j} := \{a_{i,j}, b_{i,j}, c_{i,j}\}$$

and, for each group $H_{i,j}$, let us associate to its three vertices the following sets of $k$ edges:

$$\begin{aligned}
A_{a_{i,j}} &:= \{a_{i,j}q \mid q \in V(Q)\} \\
A_{b_{i,j}} &:= \{b_{i,j}a_{i,j}\} \cup \{b_{i,j}q \mid q \in (V(Q) - \{q_1\})\} \\
A_{c_{i,j}} &:= \{c_{i,j}a_{i,j}, c_{i,j}b_{i,j}\} \cup \{c_{i,j}q \mid q \in (V(Q) - \{q_1, q_2\})\}.
\end{aligned}$$

For each $P_i$, let us define the following subset of $k$ vertices:

$$\widehat{P}_i := \{c_{i,1}, \ldots, c_{i,k}\}.$$

For each $x_i \in X := \{x_1, \ldots, x_{3t}\}$, let us define the following set of $k$ edges:

$$A_{x_i} := \left\{x_i c \;\middle|\; c \in \widehat{P}_1\right\}.$$

For each $d_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \in D$ let us define the following set of vertices:

$$D_i := \{z_{i,1}, z_{i,2}, z_{i,3}, g_{i,1}, g_{i,2}, g_{i,3}, h_{i,1}, h_{i,2}, h_{i,3}\}$$

moreover, for each vertex $z_{i,j} \in D_i$, let us define the relative set of $k$ edges:

$$\begin{aligned}
A_{z_{i,1}} &:= \left\{z_{i,1}q \;\middle|\; q \in \widehat{P}_1\right\} \\
A_{z_{i,2}} &:= \{z_{i,2}z_{i,1}\} \cup \left\{z_{i,2}q \;\middle|\; q \in (\widehat{P}_2 - \{c_{2,1}\})\right\} \\
A_{z_{i,3}} &:= \{z_{i,3}z_{i,1}, z_{i,3}z_{i,2}\} \cup \left\{z_{i,3}q \;\middle|\; q \in (\widehat{P}_3 - \{c_{3,1}, c_{3,2}\})\right\}
\end{aligned}$$

as well as for each vertex $g_{i,j} \in D_i$:

$$A_{g_{i,j}} := \{g_{i,j}z_{i,j}, g_{i,j}x_{i,j}\} \cup \left\{g_{i,j}q \;\middle|\; q \in (\widehat{P}_4 - \{c_{4,1}, c_{4,2}\})\right\}$$

and for each vertex $h_{i,j} \in D_i$:

$$A_{h_{i,j}} := \{h_{i,j}z_{i,j}, h_{i,j}x_{i,j}, h_{i,j}g_{i,j}\} \cup \left\{h_{i,j}q \;\middle|\; q \in (\widehat{P}_5 - \{c_{5,1}, c_{5,2}, c_{5,3}\})\right\}.$$

Finally, let us set:

$$A_{D_i} := A_{z_{i,1}} \cup A_{z_{i,2}} \cup A_{z_{i,3}} \cup A_{g_{i,1}} \cup A_{g_{i,2}} \cup A_{g_{i,3}} \cup A_{h_{i,1}} \cup A_{h_{i,2}} \cup A_{h_{i,3}}.$$

The graph $T$ such that:

$$\begin{aligned} V(T) &= V(Q) \cup S \cup \left(\bigcup_{i=1\dots 5} P_i\right) \cup X \cup \left(\bigcup_{i=1\dots|D|} D_i\right) \\ E(T) &= E(Q) \cup \left(\bigcup_{q_i \in S} A_{q_i}\right) \cup \left(\bigcup_{i=1\dots 5} \bigcup_{j=1\dots k} \left(A_{a_{i,j}} \cup A_{b_{i,j}} \cup A_{c_{i,j}}\right)\right) \cup \\ &\quad \cup \left(\bigcup_{i=1\dots|X|} A_{x_i}\right) \cup \left(\bigcup_{i=1\dots|D|} A_{D_i}\right) \end{aligned}$$

is a complete $k$-dense tree. In fact, if we eliminate the vertices of $T$ in an order inverted with respect to the one of the definition, that is, given $m := |D|$, if we consider the following ordering of the vertices:

$$\begin{aligned} \sigma := \quad & [h_{m,3}, h_{m,2}, h_{m,1}, g_{m,3}, g_{m,2}, g_{m,1}, z_{m,3}, z_{m,2}, z_{m,1}, \\ & \dots \\ & h_{1,3}, h_{1,2}, h_{1,1}, g_{1,3}, g_{1,2}, g_{1,1}, z_{1,3}, z_{1,2}, z_{1,1}, \\ & x_{3t}, \dots, x_1, \\ & c_{5,k}, b_{5,k}, a_{5,k}, \dots, c_{5,1}, b_{5,1}, a_{5,1}, \\ & \dots \\ & c_{1,k}, b_{1,k}, a_{1,k}, \dots, c_{1,1}, b_{1,1}, a_{1,1}, \\ & q_{k+d}, \dots, q_1] \end{aligned}$$

we obtain a $k$-elimination with properties required by Definition 3.21.

The complete $k$-dense tree $T$ has

$$n := k + d + 15k + |X| + 9|D|$$

vertices and, by Proposition 3.17, $nk - \frac{k(k+1)}{2}$ edges. Therefore, $T$ can be constructed in polynomial time in the parameters $|X|$ and $|D|$ of any instance of the problem X3C. Moreover, let us notice that $n$ is divisible by three and, in particular, we have that $n = 3b$ where:

$$b := 1 + s + 5k + t + 3|D|$$

since $|X| = 3t$ and $d = 3 - r = 3 - k + 3s$, with $s$ and $r$ the unique natural numbers such that $k = 3s + r$.

The complete $k$-dense tree $T$ is not a $k$-tree. In fact, for each integer $i$, such that $1 \leq i \leq |D|$, and for each $v \in D_i$, there is a integer $j$, with $1 \leq j \leq 5$, such that:

$$\widehat{P}_j \subseteq \Lambda^\sigma(v).$$

From the construction of $T$ it follows directly that, for each integer $l$, with $1 \leq l \leq 5$, the set of vertices $\widehat{P}_l$ is an independent set of $T$. Therefore $\Lambda^\sigma(v)$ is not a $k$-clique of $T$ and the $k$-elimination $\sigma$ does not have the properties required by Definition 3.26. Then, by Proposition 3.27, we can conclude that $T$ is not a $k$-tree.

Now, we have to prove the double stetement posed initially.
($\Leftarrow$) Let us suppose that there is a partition of $V(T)$:

$$R := \{V_1, \dots, V_b\}$$

in disjoint set such that, for each $V_i \in R$, the following holds:

$$T_{V_i} \simeq K^3.$$

For any vertex $u \in V(T)$, we denote by $V[u]$ the set of $R$ to which $u$ belongs.

Let us consider the following subset of $V(T)$:

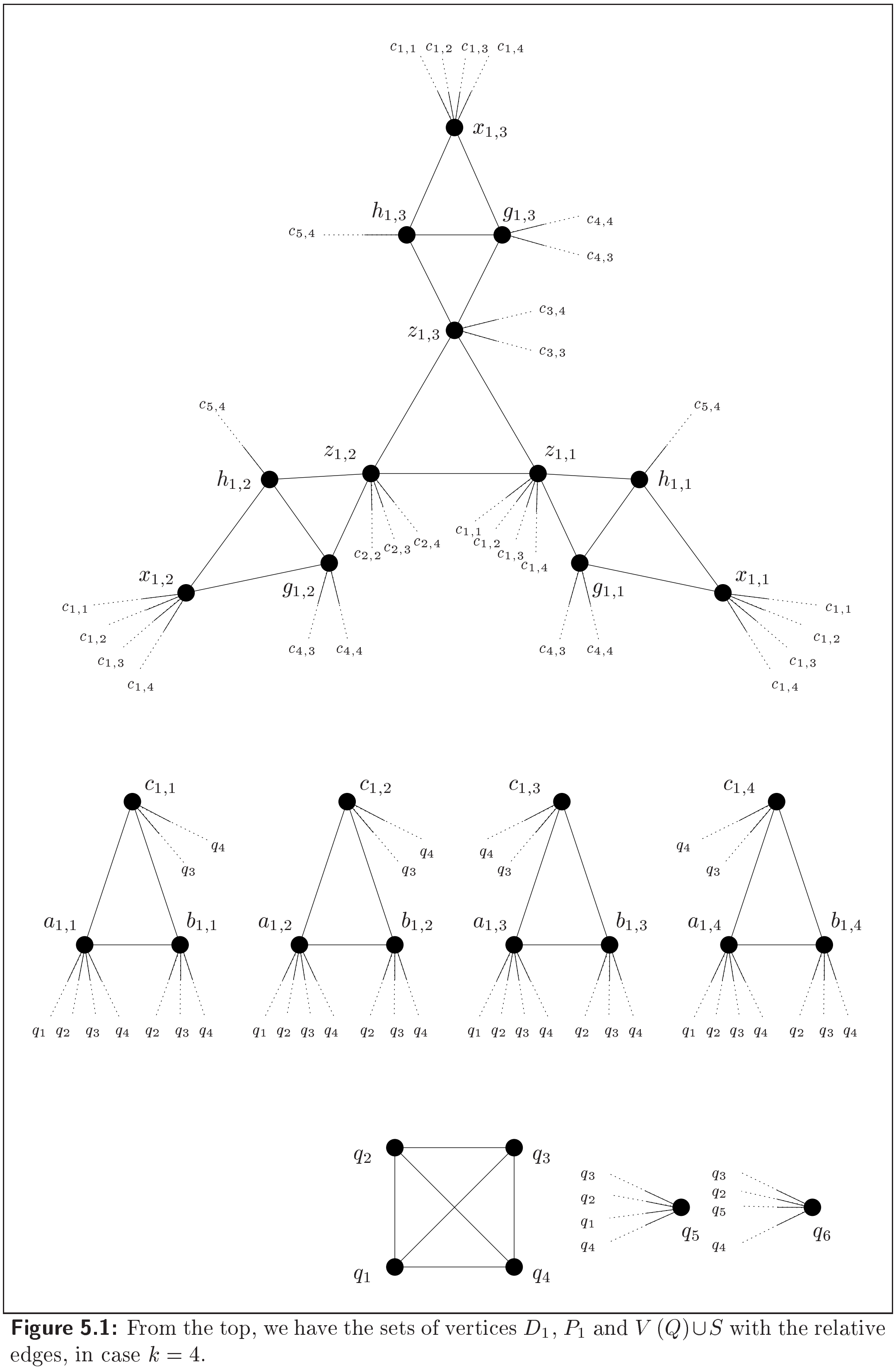

**Figure 5.1:** From the top, we have the sets of vertices $D_1$, $P_1$ and $V(Q) \cup S$ with the relative edges, in case $k = 4$.

$$W := V(Q) \cup S \cup \left( \bigcup_{i=1\ldots5} P_i \right).$$

From the definition of $T$ it follows directly that, for each edge $wu \in E(T)$ such that $w \in W$ and $u \in V(T) - W$:

$$w \in \bigcup_{i=1\ldots5} \widehat{P}_i.$$

Once again, from the definition of $T$ it follows that, for each $c \in \bigcup_{i=1\ldots5} \widehat{P}_i$, the set:

$$O_c := adj_T(c) \cap (V(T) - W)$$

is an independent set of $T$, that is $E(T_{O_c}) = \emptyset$.

Since also $\bigcup_{i=1\ldots5} \widehat{P}_i$ is an independent set of $T$, from the above considerations it follows that, for each vertex $w \in W$:

$$V[w] \subset W$$

hene, since $R$ consists of disjoint sets, also for each vertex $u \in V(T) - W$ we have:

$$V[u] \subset (V(T) - W).$$

Let us set $d_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \in D$. Let us consider the set of vertices $S_i$ which is defined in the following way:

$$S_i := D_i \cup \{x_{i,1}, x_{i,2}, x_{i,3}\}$$

it results that:

$$S_i \subseteq (V(T) - W).$$

From the definition of $T$ it follows directly that, for each $us \in E(T)$ such that:

$$s \in S_i \quad \text{and} \quad u \in V(T) - (W \cup S_i),$$

the following must holds:

$$s \in \{x_{i,1}, x_{i,2}, x_{i,3}\}.$$

From this, from the definition of $T$ and from the above considerations it follows that, for the vertices in $W$, only one of the following conditions can be verified by the vertices in $S_i$:

$$\{\{z_{i,1}, z_{i,2}, z_{i,3}\}, \{g_{i,1}, h_{i,1}, x_{i,1}\}, \{g_{i,2}, h_{i,2}, x_{i,2}\}, \{g_{i,3}, h_{i,3}, x_{i,3}\}\} \subset R \tag{1}$$

$$\{\{z_{i,1}, g_{i,1}, h_{i,1}\}, \{z_{i,2}, g_{i,2}, h_{i,2}\}, \{z_{i,3}, g_{i,3}, h_{i,3}\}\} \subset R$$

Therefore, every element of $X$ belongs to only one member of the set $D' \subseteq D$ defined in the following way:

$$D' := \{d_i \in D \mid \text{“}S_i \textit{ verifies the condition 1”}\}$$

---

$(\Rightarrow)$ Let us suppose that there is a set $D' \subseteq D$ such that every element of $X$ belongs to exactly one member of $D'$. Let us set:

$$W := V(Q) \cup S \cup \left( \bigcup_{i=1\ldots5} P_i \right).$$

The following collection of disjoint sets is a partition in triangles for the subgraph $T_{V(Q)\cup S}$:

$$R_{V(Q)\cup S} := \{\{q_1, q_2, q_3\}, \ldots, \{q_{i+1}, q_{i+2}, q_{i+3}\}, \ldots, \{q_{k+d-2}, q_{k+d-1}, q_{k+d}\}\}.$$

For each integer $i$, with $1 \leq i \leq 5$, the following collection of disjoint sets is a partition in triangles of the subgraph $T_{P_i}$:

$$R_{P_i} := \bigcup_{j=1\ldots k} \{H_{i,j}\}.$$

Hence, the collection of disjoint sets:

$$R' := R_{V(Q)\cup S} \cup \left( \bigcup_{i=1\ldots 5} R_{P_i} \right)$$

is a partition in triangles for the subgraph $T_W$.

For each $d_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \in D$, let us define $R_{d_i}$, a collection of sets of vertices associated to $d_i$, in the following way:

$$R_{d_i} := \begin{cases} \{\{z_{i,1}, z_{i,2}, z_{i,3}\}, \{g_{i,1}, h_{i,1}, x_{i,1}\}, \{g_{i,2}, h_{i,2}, x_{i,2}\}, \{g_{i,3}, h_{i,3}, x_{i,3}\}\} & d_i \in D' \\ \{\{z_{i,1}, g_{i,1}, h_{i,1}\}, \{z_{i,2}, g_{i,2}, h_{i,2}\}, \{z_{i,3}, g_{i,3}, h_{i,3}\}\} & d_i \notin D' \end{cases}$$

By the hypothesis on $D'$, the collection of disjoint sets of vertices:

$$R'' := \bigcup_{d_i \in D} R_{d_i}$$

is a partition in triangles for the subgraph $T_{V(T)-W}$.

Obviously, we have that $V(T_W) \cap V\left(T_{V(T)-W}\right) = \emptyset$ and so we can conclude that the collection of disjoint sets $R := R' \cup R''$ is a partition in triangles for the complete $k$-dense tree $T$. $\square$

# 6 Dominating Set

**Problem 6.1 (Dominating Set or DS)**
INSTANCE: *A graph $G$ and a positive integer $H \leq |G|$.*
QUESTION: *Is there a dominating set of cardinality less or equal to $H$? That is, is there a set of vertices $V' \subseteq V(G)$ with $|V'| \leq H$ and such that, for each $u \in V(G) - V'$, there is an edge $uv \in E(G)$ with $v \in V'$?*

**Theorem 6.2**
*For $k \geq 2$, the problem Dominating Set restricted to complete $k$-dense trees still is* NP.

Proof. DS is an NP-complete problem, hence its version restricted to complete $k$-dense trees belongs to NP-complete.

We will define a transformation from 3-SAT to the problem DS restricted to complete $k$-dense trees. Therefore, given a set $U := \{u_1, \ldots, u_n\}$ of Boolean variables and given a set: $C := \{c_1, \ldots, c_m\}$ of clauses such that $c_i \subseteq U$ and $|c_i| = 3$ for all $1 \leq i \leq m$, we will construct a complete $k$-dense tree $T$ and we will define a positive integer $H \leq |T|$ such that:

> "there is a truth assignment for $U$ which satisfies all the clauses in $C$"
>
> $$\Longleftrightarrow$$
>
> "there is a set of vertices $V' \subseteq V(T)$ with $|V'| \leq H$ and such that, for each $u \in V(T) - V'$, there is an edge $uv \in E(T)$ with $v \in V'$."

Let $Q$ a complete graph of order $k$ with $V(Q) := \{q_1, \ldots, q_k\}$. Given $t := 6m + n$, for each $i$, with $1 \leq i \leq t$, let us define the set of $k-2$ vertices:

$$R_i := \{r_{i,1}, \ldots, r_{i,k-2}\}$$

and, for each $r_{i,j} \in R_i$, let us define the set of $k$ edges:

$$A_{r_{i,j}} := \{r_{i,j}q \mid q \in V(Q)\}.$$

Moreover, let us set:

$$R := \bigcup_{i=1\ldots t} R_i$$

and

$$A_R := \bigcup_{i=1\ldots t} \left( \bigcup_{j=1\ldots k-2} A_{r_{i,j}} \right).$$

Similarly, for each $i$, with $1 \leq i \leq t$, let us define the set of $k-2$ vertices:

$$F_i := \{f_{i,1}, \ldots, f_{i,k-2}\}$$

and, for each $f_{i,j} \in F_i$, let us define the set of $k$ edges:

$$A_{f_{i,j}} := \{f_{i,j}q \mid q \in V(Q)\}.$$

Finally, let us set:

$$F := \bigcup_{i=1\ldots t} F_i$$

and

$$A_F := \bigcup_{i=1\ldots t} \left( \bigcup_{j=1\ldots k-2} A_{f_{i,j}} \right).$$

Let us define the set of $2n$ vertices:

$$L := \{l \mid l \in \{u_1, \ldots, u_n\} \cup \{\overline{u}_1, \ldots, \overline{u}_n\}\}.$$

and, for each vertex $l \in L$, let us define the following set of $k$ edges:

$$A_l := \{lq \mid q \in V(Q)\}.$$

For each clause $c_i \in C$, let us define the set of three vertices:

$$S_i := \{s_{i,1}, s_{i,2}, s_{i,3}\}$$

and, for each vertex $s_{i,j} \in S_i$, let us define the following set of $k$ edges:

$$A_{s_{i,j}} := \{s_{i,j}q \mid q \in V(Q)\}.$$

Let us define the following collections of sets of two vertices:

$$\begin{aligned}
D_{\{L\}} &:= \{\{u_i, \overline{u}_i\} \mid 1 \leq i \leq n\} \\
D_{\{S\}} &:= \{\{s_{i,1}, s_{i,2}\}, \{s_{i,1}, s_{i,3}\}, \{s_{i,2}, s_{i,3}\} \mid 1 \leq i \leq m\} \\
D_{\{S,L\}} &:= \{\{s_{i,1}, l_{i,1}\}, \{s_{i,2}, l_{i,2}\}, \{s_{i,3}, l_{i,3}\} \mid 1 \leq i \leq m \wedge c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}\}
\end{aligned}$$

and let us set:

$$D := D_{\{L\}} \cup D_{\{S\}} \cup D_{\{S,L\}}.$$

Therefore, we have that $|D| = 6m + n = t$. Let us suppose that the members of $D$ are ordered in an arbitrary way and, for each $d_i = \{d_{i,1}, d_{i,2}\} \in D$, let us define the set of two vertices:

$$P_i := \{p_{i,1}, p_{i,2}\}$$

and, for the vertices of $P_i$, let us define the following sets of $k$ edges:

$$A_{p_{i,1}} := \{p_{i,1}d_{i,1}, p_{i,1}d_{i,2}\} \cup \{p_{i,1}r \mid r \in R_i\}$$

$$A_{p_{i,2}} := \{p_{i,2}d_{i,1}, p_{i,2}d_{i,2}\} \cup \{p_{i,2}f \mid f \in F_i\}$$

moreover, let us set:

$$P := \bigcup_{i=1\ldots t} P_i.$$

The graph $T$ such that:

$$\begin{aligned} V(T) &= V(Q) \cup R \cup F \cup L \cup \left(\bigcup_{i=1\ldots m} S_i\right) \cup \left(\bigcup_{i=1\ldots t} P_i\right) \\ E(T) &= E(Q) \cup A_R \cup A_F \cup \left(\bigcup_{l\in L} A_l\right) \cup \left(\bigcup_{i=1\ldots m} (A_{s_{i,1}} \cup A_{s_{i,2}} \cup A_{s_{i,3}})\right) \cup \\ &\quad \cup \left(\bigcup_{i=1\ldots t} (A_{p_{i,1}} \cup A_{p_{i,2}})\right) \end{aligned}$$

is a complete $k$-dense tree. In fact, if we eliminate the vertices of $T$ in inverted order with respect to that one of definition, that is if we consider the following ordering of the vertices:

$$\begin{aligned} \sigma := \quad & [p_{t,2}, p_{t,1}, \\ & \ldots \\ & p_{1,2}, p_{1,1}, \\ & s_{m,3}, s_{m,2}, s_{m,1}, \\ & \ldots \\ & s_{1,3}, s_{1,2}, s_{1,1}, \\ & \overline{u}_n, \ldots, \overline{u}_1, \\ & u_n, \ldots, u_1, \\ & f_{t,k-2}, \ldots, f_{t,1}, \\ & \ldots \\ & f_{1,k-2}, \ldots, f_{1,1}, \\ & r_{t,k-2}, \ldots, r_{t,1}, \\ & \ldots \\ & r_{1,k-2}, \ldots, r_{1,1}, \\ & q_k, \ldots, q_1] \end{aligned}$$

we obtain a $k$-elimination with the properties required by Definition 3.21.

In order to complete the transformation, let us set:

$$H := 2m + n + 1.$$

The complete $k$-dense tree $T$ has $w := k + 2t(k-2) + 2n + 3m + 2t$ vertices and, by Proposition 3.17, $wk - \frac{k(k+1)}{2}$ edges. Then, since $t = 6m + n$, $T$ can be constructed in polynomial time with respect to the parameters $n$ and $m$ of any instance of the problem 3-SAT.

The complete $k$-dense tree $T$ is not a $k$-tree. In fact, for each integer $i$, with $1 \le i \le t$, the set $\Lambda^\sigma(p_{i,1})$ (as well as $\Lambda^\sigma(p_{i,2})$) is an independent set of $T$, hence the $k$-elimination $\sigma$ does not have the properties required by Definition 3.26. Then, by Proposition 3.27, we can conclude that $T$ is not a $k$-tree.

In order to complete the proof, we have to verify the double implication that we have initially introduced.

($\Leftarrow$) Let us suppose that there is a set of vertices $V' \subseteq V(T)$ with $|V'| \le H$ and such that, for each $u \in V(T) - V'$, there is an edge $uv \in E(T)$ with $v \in V'$. We will highlight a fundamental property of $T$.

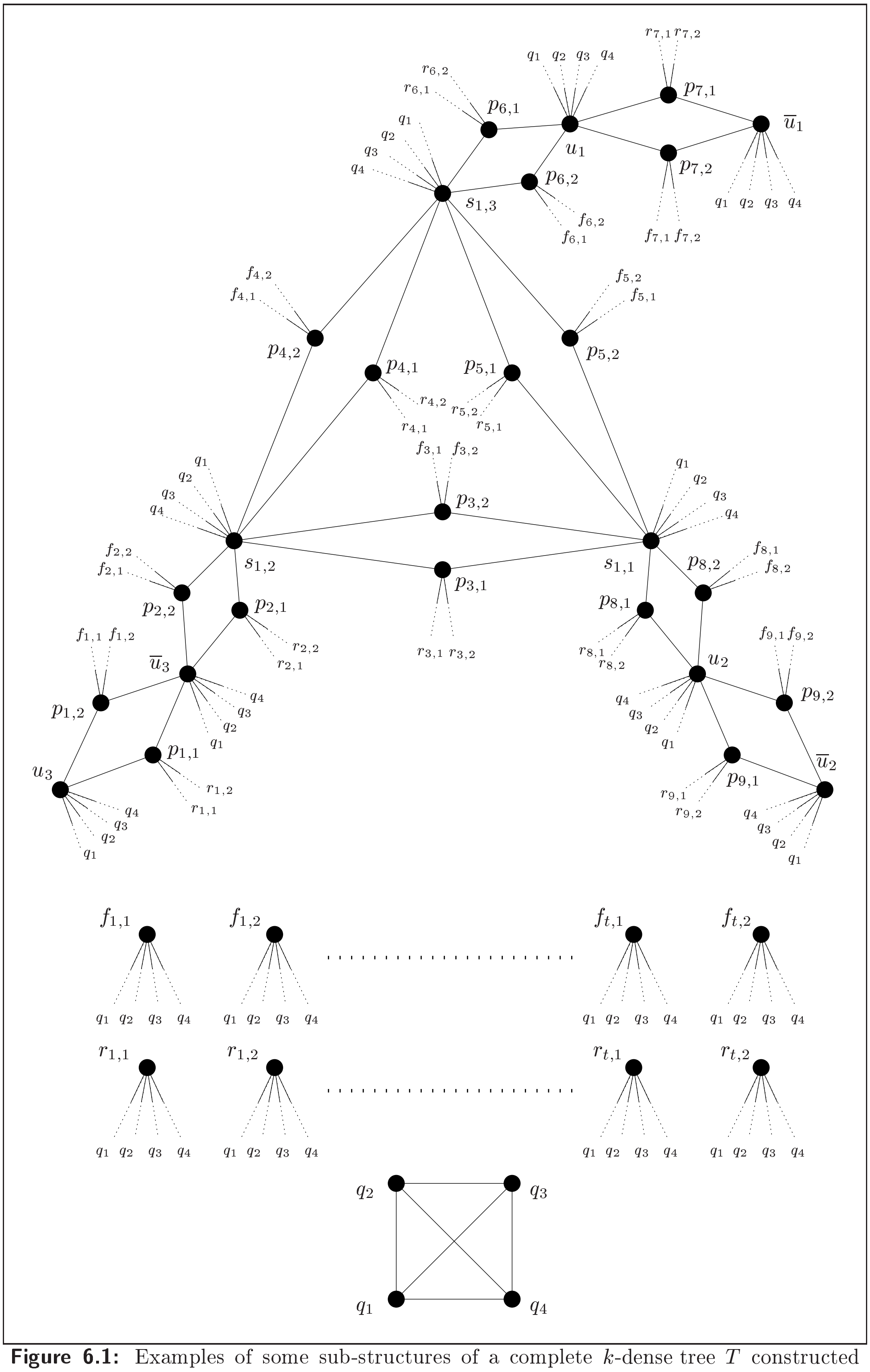


**Figure 6.1:** Examples of some sub-structures of a complete $k$-dense tree $T$ constructed following the proof of Theorem 6.2 (for $k = 4$). In particular, the subgraph of $T$ relative to the clause $c_1 = \{u_2, \overline{u}_3, u_1\}$ is represented in the upper part.

*If $V \subset V(T)$ is a dominating set for $T$ such that $|V| \leq H$ then there is a vertex $q \in V(Q)$ which is included in $V$.*

—————————

Let us suppose that the following holds:

$$V \cap V(Q) = \emptyset.$$

Since $V$ is a dominating set for $T$, for each integer $i$ with $1 \leq i \leq t$, only one of the following eventualities can happen for the set $R_i$:

(i) $p_{i,1} \in V$,

(ii) $p_{i,1} \notin V$ and $R_i \subset V$.

In fact, if $p_{i,1} \notin V$ and there was a vertex $r' \in R_i$ which was not included in $V$ then, since $V$ does not contain vertices of $V(Q)$ (by hypothesis) and since, for each $r \in R_i$, we have:

$$adj_T(r) = \{p_{i,1}\} \cup V(Q),$$

there would not be any edge $r'v \in E(T)$ with $v \in V$ and so $V$ would not be a dominating set for $T$.

Therefore, for each integer $j$ with $1 \leq j \leq t$, there is at least one vertex $v_j \in \{p_{i,1}\} \cup R_j$ which is not included in $V$. Since for any two integer $y, z$ with $1 \leq y < z \leq t$, we have:

$$(\{p_{y,1}\} \cup R_y) \cap (\{p_{z,1}\} \cup R_z) = \emptyset$$

it is clear that:

$$|V| \geq |\{v_j \mid 1 \leq j \leq t\}| = t = 6m + n > H = 2m + n + 1,$$

against the hypothesis on the cardinality of $V$.

Then there is a vertex $q \in (V(Q) \cap V)$. Let us set:

$$X := \{\{u_i, \overline{u}_i\} \mid 1 \leq i \leq n\} \cup \{S_j \mid 1 \leq j \leq m\},$$

and let us prove another crucial property of $T$.

*Let $V \subset V(T)$ be a dominating set for $T$ with $|V| \leq H$. For each $x \in X$, the followings hold:*

*(i) if $x = \{u_i, \overline{u}_i\}$ then $|x \cap V| = 1$,*

*(ii) if $x = S_i$ then $|x \cap V| = 2$, and no vertex of $R$, $F$ and $P$ belongs to $V$.*

—————————

For each $x \in X$, let us set:

$$P_x := \{p_{i,j} \in P \mid |adj_T(p_{i,j}) \cap x| = 2\}$$

$$Dom(P_x) := P_x \cup \left(\bigcup_{p \in P_x} adj_T(p)\right)$$

from the definition of $P_x$ it follows that:

$$x \subset Dom(P_x).$$

For each $x \in X$, let us consider the collection $\mathbb{I}_x$ of the minimal sets $I$ satisfying the following properties:

- $I \subseteq Dom\,(P_x)$,
- for each vertex $u \in (P_x - I)$ there is an edge $uv \in E\,(T)$ with $v \in I$.

Hence, if $V'' \subseteq V\,(T)$ is a dominating set for $T$ then, for each $x \in X$, there must be a set $I \in \mathbb{I}_x$ which is included in $V''$; in fact, if there was $x' \in X$ such that, for each set $I'$ in $\mathbb{I}_x$:

$$I' \nsubseteq V''$$

then, by the minimality of the sets in $\mathbb{I}_x$, there would be a vertex $p \in P_x$ with $p \notin V''$ and such that:

$$adj_T\,(p) \cap V'' = \emptyset,$$

against the hypothesis that $V''$ is a dominating set for $T$.

For a generic $x \in X$, let us consider the possible cardinalities of a set $I \in \mathbb{I}_x$. We must distinguish two cases.

- If there is an integer $i$ such that:

  $$x = \{u_i, \overline{u}_i\}$$

  then, for a set $I \in \mathbb{I}_x$, we can have:
  - $|I| = 1$, if $I = \{u\}$ with $u \in \{u_i, \overline{u}_i\}$.
  - $|I| = 2$, if $I \cap \{u_i, \overline{u}_i\} = \emptyset$.
- Otherwise, if there is an integer $i$ such that:

  $$x = S_i = \{s_{i,1}, s_{i,2}, s_{i,3}\}$$

  then, for a set $I \in \mathbb{I}_x$, we can have:
  - $|I| = 2$, if $I \subset \{s_{i,1}, s_{i,2}, s_{i,3}\}$,
  - $|I| \geq 3$, if $I \not\subset \{s_{i,1}, s_{i,2}, s_{i,3}\}$.

Since $V$ is a dominating set for $T$, for what we have written previously, we know that, for each $x \in X$, there is a set $I_x \in \mathbb{I}_x$ such that $I_x \subseteq V$.

Moreover, for each $x', x'' \in X$, it follows directly from the definition of $T$ that:

$$Dom\,(P_{x'}) \cap Dom\,(P_{x''}) = \emptyset$$

hence, for any two sets $I' \in \mathbb{I}_{x'}$ and $I'' \in \mathbb{I}_{x''}$, we have:

$$I' \cap I'' = \emptyset.$$

Consequently, we have:

$$|V| \geq \left| \bigcup_{x \in X} I_x \right| \geq 2m + n.$$

Moreover, since $V \leq H = 2m + n + 1$, we know that there is a vertex $q \in V\,(Q)$ in $V$ hence we can conclude that:

$$|V| = H = 2m + n + 1.$$

Therefore, it is clear that, for each $x \in X$, only two cases occur for the set $I_x$:

- $|I_x| = 1$, if there is an integer $i$ such that:

  $$x = \{u_i, \overline{u}_i\},$$

- $|I_x| = 2$, if there is an integer $i$ such that:

  $$x = S_i = \{s_{i,1}, s_{i,2}, s_{i,3}\}.$$

Therefore, the property is proved.

By the above property, we know that the assignment function $\mathcal{S} : U \to \{True, False\}$ such that, for each $u_i \in U$:

$$\mathcal{S}(u_i) := \begin{cases} True & u_i \in V' \\ \\ False & \overline{u}_i \in V' \end{cases}$$

is well-defined.

Now, we have to prove that all the clauses in $C$ are satisfied by $\mathcal{S}$. Let $c_z \in C$. Since $V'$ is a dominating set for $T$ with $|V'| \leq H$, we know that, for each vertex $p \in P$ which satisfies the following conditions:

$$\begin{aligned} c_z \cap adj_T(p) &= \{\widehat{u}\} \neq \emptyset \\ \\ S_z \cap adj_T(p) &= \{\widehat{s}\} \neq \emptyset \end{aligned}$$

at least one of the vertices $\widehat{u}$ and $\widehat{s}$ must belong to $V'$.

By the second property of $T$, we know that only two vertices of $S_z$ belong to $V'$ and that no vertex of $R$, $F$ or $P$ is contained in $V'$. Therefore, if $s'$ is the vertex of $S_z$ which does not belong to $V'$ and $d := \{s', l'\}$ is the set of two vertices which belongs to $D_{\{S,L\}}$, then the vertex $l'$ must be contained in $V'$. Consequently, for the definition of $\mathcal{S}$, the literal $l'$ is satisfied, hence is the clause $c_j$ containing it.

---

($\Rightarrow$) Let $\mathcal{S}_U : U \to \{True, False\}$ be an assignment for the Boolean variables in $U$ satisfying all the clauses in $C$. Let us define the following set of $n$ vertices:

$$V_U := \{l \mid l \in L \wedge \text{ “\textit{the literal} } l \textit{ is satisfied by } \mathcal{S}_U\text{”}\}.$$

For each clause $c_j \in C$, let us choose a vertex $s_j \in S_j$ such that:

$$adj_T(s_j) \cap V_U \neq \emptyset$$

(obviously, a vertex with these characteristics exists, because $\mathcal{S}_U$ satisfies all the clauses of $C$) and let us define the set of $2m$ vertices:

$$V_C := \bigcup_{i=1\ldots m} S_i - \{s_i\}.$$

Finally, let $q$ be a vertex in $V(Q)$. We will prove that the set of $2m + n + 1$ vertices:

$$V'' := V_U \cup V_C \cup \{q\}$$

is a dominating set for $T$, and so that for each $u \in V(T) - V''$ there is an edge $uv \in E(T)$ with $v \in V''$.

Since the vertex $q \in V(Q)$ belongs to $V''$, it results that, for each vertex $u$ belonging to one of the following sets:

- $V(Q) - \{q\}$,
- $R$,
- $F$,
- $L - V''$,
- $S_i - V''$, for each integer $i$ with $1 \leq i \leq m$,

there is an edge $uv \in E(T)$ with $v \in V''$ $(v = q)$.

Now, we have to verify the vertices in $P$. Since, for each integer $i$ with $1 \leq i \leq n$, one of the vertices in $\{u_i, \overline{u}_i\}$ belongs to $V''$, then, for each $p \in P$ such that:

$$|adj_T(p) \cap L| = 2$$

(that is, such that there is a set of two vertices $d_L \in D_{\{L\}}$ with $d_L \subseteq adj_T(p)$), there is an edge $pv \in E(T)$ with $v \in V''$.

Similarly, since for each integer $i$ with $1 \leq i \leq m$, the two vertices in $S_i - \{s_i\}$ are included in $V''$, we have that:

- for each $p \in P$ for which there is an integer $j$, with $1 \leq j \leq m$, such that:

$$|adj_T(p) \cap S_j| = 2$$

  (that is, for which there is a set of two vertices $d_S \in D_{\{S\}}$ such that $d_S \subseteq adj_T(p)$), there is an edge $pv \in E(T)$ with $v \in V''$,

- for each $p \in P$ for which there are an integer $j$, with $1 \leq j \leq m$, and a set of two vertices $d \in D_{\{S,L\}}$ such that:

$$d \subseteq adj_T(p) \quad \text{and} \quad (S_j - \{s_j\}) \cap d \neq \emptyset,$$

  there is an edge $pv \in E(T)$ with $v \in V''$.

Finally, for each integer $j$, with $1 \leq j \leq m$, we have to verify the two vertices $p'$ and $p''$ of $P$ adjacent to the vertex $s_j$ and to a vertex in $L$. By the choice of this vertex, we know that the literal corresponding to $s_j$ in the clause $c_j$ is satisfied by $\mathcal{S}_U$ hence, the vertex of $L$ corresponding to this literal, and adjacent to $p'$ and $p''$, is included in $V''$. □

# 7 Graph Colorability

**Problem 7.1 (Graph Colorability or GC** [Kar72]**)**
INSTANCE: *A graph $G$ and a positive integer $r \leq |T|$.*
QUESTION: *Is the graph $G$ $r$-colourable? That is, is there a function*

$$c : V(G) \to \{1, \ldots, r\}$$

*such that $c(v) \neq c(u)$ if $v$ and $u$ are adjacent?*

By Proposition 3.25, we know that for each non-trivial complete $k$-dense tree $T$, the following holds:

$$\chi(T) = k + 1.$$

Then, the Problem 7.1 restricted to complete $k$-dense trees is no longer NP-complete; moreover it is solvable in constant time, since, for each complete $k$-dense tree $T$:

- if $T$ is trivial, that is $|T| \leq k$, then is a complete graph hence, the problem GC on $T$ answers true if and only if $r = |T|$,
- if $T$ is non-trivial, the problem GC on $T$ answers true if and only if $r \geq \chi(T) = k + 1$.

Therefore, the following result holds.

**Theorem 7.2**
*For each integer $k \geq 1$, the problem GC restricted to complete $k$-dense trees can be solved in constant time.*

From the proof of Proposition 3.18 (see [Fra01a]) it is easy to draw a simple algorithm for finding in linear time a $(k+1)$-colouring for any $k$-dense forest by a $k$-elimination.

# 8 Disjoint Connecting Paths

**Problem 8.1 (Disjoint Connecting Paths o DCP** [Kar75]**)**
INSTANCE: *A graph $G$, an integer $d$ and a collection $C$ of $d$ disjoint sets of two vertices of $G$*

$$C := \{\{s_1, z_1\}, \ldots, \{s_d, z_d\}\}.$$

QUESTION: *Are there in $G$ $d$ disjoint paths $P_1, \ldots, P_d$ such that, for each integer $i$ with $1 \leq i \leq d$, $P_i = s_i \ldots z_i$?*

We know (see [Fra01a] or [Fra01b]) that any complete $k$-dense tree $T$ with $|T| \geq 2\left\lfloor\frac{k+1}{2}\right\rfloor$, is $r$-linked, where $r := \left\lfloor\frac{k+1}{2}\right\rfloor$, and that $r$ is the greatest integer for which this is possible. Therefore, for *any* collection of $r$ disjoint sets of two vertices $\{\{s_1, z_1\}, \ldots, \{s_r, z_r\}\}$, we know that $r$ disjoint paths $P_1, \ldots, P_r$ exist in $T$ such that $P_i = s_i \ldots z_i$, for all $1 \leq i \leq r$.

Therefore, if $d \leq r = \left\lfloor\frac{k+1}{2}\right\rfloor$, the problem DCP restricted to complete $k$-dense trees has always a positive answer. Then, the following result holds.

**Theorem 8.2**
*For each integer $k \geq 1$, the problem DCP restricted to complete $k$-dense trees can be solved in constant time if $d \leq r = \left\lfloor\frac{k+1}{2}\right\rfloor$.*

Moreover, from the proof of the "$\left\lfloor\frac{k+1}{2}\right\rfloor$-linkability" of a complete $k$-dense tree we are able to draw a recursive algorithm that finds the required disjoint paths.

---

**Algorithm 8.1** Location of $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths among $\left\lfloor\frac{k+1}{2}\right\rfloor$ pairs of vertices of a complete $k$-dense tree

---

**Input:** $\Rightarrow$ $T$, complete $k$-dense tree,

$\Rightarrow$ $\sigma$, $k$-elimination for $T$,

$\Rightarrow$ $S$ and $Z$, sets of $\left\lfloor\frac{k+1}{2}\right\rfloor$ vertices of $T$ such that $S \cap Z = \emptyset$,

$\Rightarrow$ $C$, array of $\left\lfloor\frac{k+1}{2}\right\rfloor$ sets of vertices such that

- for each $l$, $1 \leq l \leq \left\lfloor\frac{k+1}{2}\right\rfloor$, $C[l] = \{s_l, z_l\}$ with $s_l \in S$ and $z_l \in Z$,
- for each $1 \leq i < j \leq \left\lfloor\frac{k+1}{2}\right\rfloor$, $C[i] \cap C[j] = \emptyset$.

**Output:** $\Leftarrow$ $A$, array of $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths among the vertices of the sets of $C$.

**Data:** - $B$, array of $\left\lfloor\frac{k+1}{2}\right\rfloor$ Boolean variables.

1: **for** $i = 1$ to $\left\lfloor\frac{k+1}{2}\right\rfloor$ **do**
2: $\quad B[i] \leftarrow False$
3: **end for**
4: **return** FindDisjointPaths$(T, \sigma, S, Z, C, B)$

---

**Theorem 8.3**
*Let $T$ be a complete $k$-dense tree such that $|T| \geq 2\left\lfloor\frac{k+1}{2}\right\rfloor$ and let $\sigma$ be a $k$-elimination for $T$. Let $S$ and $Z$ be disjoint sets of $\left\lfloor\frac{k+1}{2}\right\rfloor$ vertices of $T$. Let $C$ be an array of $\left\lfloor\frac{k+1}{2}\right\rfloor$ sets of vertices such that:*

*(i) for each $l$, $1 \leq l \leq \left\lfloor\frac{k+1}{2}\right\rfloor$, $C[l] = \{s_l, z_l\}$, with $s_l \in S$ and $z_l \in Z$,*

*(ii) for each $i, j$, $1 \leq i < j \leq \left\lfloor\frac{k+1}{2}\right\rfloor$, $C[i] \cap C[j] = \emptyset$.*

*The Algorithm 8.1, applied to $T$, $\sigma$, $S$, $Z$ and $C$, returns an array $A$ of $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths such that, for each integer $i$, $A[i]$ link the two vertices of $C[i]$.*

Proof. In Algorithm 8.1 the array $B$ reflects the initial situation, where none of the searched paths is known. Subsequently, it calls the Function 8.2 and returns the array obtained by Function 8.2. We have to prove that Function 8.2, applied to $B$ and to the data structures which Algorithm 8.1 received as input, returns required array $A$. The Function 8.2 is defined by recursive calls on a complete $k$-dense tree, obtained by deleting a $k$-leaf of $T$. Then, it is natural that we prove the thesis by induction on $|T|$.

Let us suppose that $|T| = n = 2\left\lfloor\frac{k+1}{2}\right\rfloor$. If $k$ is even then $\left\lfloor\frac{k+1}{2}\right\rfloor = \frac{k}{2}$ and $n = k$. Conversely, if $k$ is odd, $\left\lfloor\frac{k+1}{2}\right\rfloor = \frac{k+1}{2}$ and $n = k+1$. However, in both cases $T \simeq K^n$ hence, for each set $C[i] := \{s_i, z_i\}$, the searched path is $P_i := s_i z_i$ (cycle of step 4).

**Function 8.2** FindDisjointPaths

**Input:**

- $\Rightarrow$ $T$, complete $k$-dense tree.
- $\Rightarrow$ $\sigma$, $k$-elimination for $T$.
- $\Rightarrow$ $S$ and $Z$, sets of $\left\lfloor \frac{k+1}{2} \right\rfloor$ vertices of $T$ such that $S \cap Z = \emptyset$.
- $\Rightarrow$ $C$, array of $\left\lfloor \frac{k+1}{2} \right\rfloor$ sets of vertices such that
  - for each $l$, $1 \leq l \leq \left\lfloor \frac{k+1}{2} \right\rfloor$, $C[l] = \{s_l, z_l\}$ with $s_l \in S$ and $z_l \in Z$,
  - for each $1 \leq i < j \leq \left\lfloor \frac{k+1}{2} \right\rfloor$, $C[i] \cap C[j] = \emptyset$,
- $\Rightarrow$ $B$, array of $\left\lfloor \frac{k+1}{2} \right\rfloor$ Boolean variables.

**Output:**

- $\Leftarrow$ $A$, array of $\left\lfloor \frac{k+1}{2} \right\rfloor$ disjoint paths among the vertices of the sets of $C$.

```
if ∃i.B[i] = False then
  r ← ⌊(k+1)/2⌋
  if |T| = 2r then
    for all {s_i, z_i} in C do
      A[i] ← s_i z_i
    end for
  else
    f ← σ(1)
    σ' ← tail(σ)
    if f ∉ S ∧ f ∉ Z then
      A ← FindDisjointPaths(T_{V(T)−{f}}, σ', S, Z, C, B)
    else
      l ← ⟨d ∈ V(T) such that {f, d} ∈ C⟩
      i ← ⟨index in C of the set I such that f ∈ I⟩
      if l ∈ adj_T(f) then
        B[i] ← True
        h ← ⟨any vertex in V(T) − (S ∪ Z)⟩
        C[i] ← {h, l}
        if f ∈ S then
          S' ← S − {f} ∪ {h}; Z' ← Z
        else
          Z' ← Z − {f} ∪ {h}; S' ← S
        end if
        A ← FindDisjointPaths(T_{V(T)−{f}}, σ', S', Z', C, B)
        A[i] ← fl
      else
        v ← ⟨any vertex in adj_T(f) − (S ∪ Z)⟩
        C[i] ← {v, l}
        if f ∈ S then
          S'' ← S − {f} ∪ {v}; Z'' ← Z
        else
          Z'' ← Z − {f} ∪ {v}; S'' ← S
        end if
        A ← FindsDisjointPaths(T_{V(T)−{f}}, σ', S'', Z'', C, B)
        A[i] ← fvA[i]
      end if
    end if
  end if
end if
return A
```

If $|T| = n > 2\left\lfloor\frac{k+1}{2}\right\rfloor$, let us suppose that for each complete $k$-dense tree of order $n-1$ and for each collection of $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint pairs of its vertices, Function 8.2 finds the required disjoint paths. The vertex $f$ of $T$ selected at step 8 is a $k$-leaf of $T$ hence, the subgraph $T' := T_{V(T)-\{f\}}$ used in the recursive calls inside Function 8.2, is a complete $k$-dense tree of order $n-1$. We have two cases.

- $f \notin S$ and $f \notin Z$, that is, the condition of step 10 is true. Then, by the inductive hypothesis, if we apply (step 11) Function 8.2 to $T'$, $\sigma'$ (defined at step 9), $S$, $Z$, $C$ and $B$, we obtain an array $A$ of disjoint paths of $T'$ such that, for each $i$ with $1 \le i \le \left\lfloor\frac{k+1}{2}\right\rfloor$, $A[i]$ links the two vertices of $C[i]$. All the paths contained in $A$ also maintain their characteristics in $T$, then the execution stops correctly.

- $f \in S$ or $f \in Z$, that is, the condition of step 10 is not true. Let $l$ be the vertex in $S \cup Z$ paired with $f$ (step 13) and let $i$ be the index in $C$ of the set $\{f, l\}$ (step 14), we have two cases.

  - $l \in adj_T(f)$.
    In this case, the path $fl$ is obviously a path between $f$ and its "partner", therefore we can set to $True$ the Boolean variable $B[i]$ (step 16). The choice of a vertex $h$ in $V(T) - (S \cup Z)$ (step 17) paired with $l$ (step 18) and the changes made in the set of vertices ($S$ or $Z$) to which $f$ belongs (step $19, \ldots, 23$), show that among the $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths that, by inductive hypothesis, are returned in $A$ by the recursive call of FindsDisjointPaths of step 24, only the one of index $i$ contains the vertex $l$. Therefore, if we replace in $A[i]$ the path between $h$ and $l$ with the one of length 1 between $f$ and $l$ (step 25), we obtain $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths among the vertices of the sets of array $C$.
  - $l \notin adj_T(f)$.
    In this case, since $|S| = |Z| = \left\lfloor\frac{k+1}{2}\right\rfloor$, the set $adj_T(f) - (S \cup Z)$ is non-empty. If we choose a vertex $v$ in $adj_T(f) - (S \cup Z)$ (step 27) and if we replace $f$ with it in $C[i]$ (step 28) and in the set ($S$ or $Z$) to which $f$ belongs (steps $29, \ldots, 33$), by the inductive hypothesis, we know that the recursive call of step 34 returns an array $A$ of $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths which, obviously, don't contain $f$. Therefore the path $fvA[i]$ links $f$ and $l$ and it is disjoint from the other paths in $A$. Consequently, by the assignment at step 35, we obtain $\left\lfloor\frac{k+1}{2}\right\rfloor$ disjoint paths among the vertices of the sets of array $C$. □

**Theorem 8.4**
*Algorithm 8.1, applied to a complete $k$-dense tree $T$ of order $n$ and to a $k$-elimination for $T$, is of $O(kn)$ time complexity.*

PROOF. Trivial. □

Finally, if $d > r$, we can affirm only two things:

- since $r = \left\lfloor\frac{k+1}{2}\right\rfloor$ is the greatest integer $p$ for which a complete $k$-dense tree can be $p$-linked, for each complete $k$-dense tree there must be at least one collection of $d$ disjoint sets for which the problem DCP has a negative answer (for instance, a collection of $k+1$ disjoint sets such that there is a set containing a $k$-leaf and each one of the other sets contains one of the $k$ vertices adjacent to the $k$-leaf).

- for each integer $d'$, there is an instance of the problem DCP such that:

  - $G$ is a complete $k$-dense tree,
  - $|C| = d'$

  giving answer true. For example, let $T$ be a complete $k$-dense tree with $|T| \ge 2d'$ and let $C$ be a collection of $d'$ disjoint sets of two vertices of $T$; if we add to $T$ $d'$ $k$-leaves such that each of these new vertices is adjacent to the two vertices of a member of $C$ (and to $k-2$ other vertices), we obtain a complete $k$-dense tree $T'$ with $d'$ disjoint paths (of length 2) linking the vertices of the members of $C$.

# 9 Monochromatic Triangle

**Problem 9.1 (Monochromatic Triangle o MT)**
INSTANCE*: A graph $G$.*
QUESTION*: Is there a partition of $E(G)$ in two disjoint sets $A_1$, $A_2$ such that neither $G_1 := (V(G), A_1)$ nor $G_2 := (V(G), A_2)$ has $K^3$ as subgraph?*

**Proposition 9.2**
*For each 4-dense forest $F$ there is a partition of $E(F)$ in two disjoint sets $A_1$, $A_2$ such that:*

*(i) neither $F_1 := (V(F), A_1)$ nor $F_2 := (V(F), A_2)$ has $K^3$ as subgraph,*

*(ii) for each 4-clique $D$ of $F$, the graphs $(D, E(F_D) \cap A')$ and $(D, E(F_D) \cap A'')$ are both paths (of four vertices).*

PROOF. If $|F| \leq 3$, the thesis is obvious. Conversely, if $|F| \geq 4$, we will prove the thesis by induction on $|F|$.

If $|F| = 4$, the only significant is the one in which $F$ is a complete graph: let $v_1, v_2, v_3, v_4$ be the vertices of $F$; the partition of the edges of $F$ in the two disjoint sets $\{v_1v_2, v_2v_3, v_3v_4\}$ and $\{v_3v_1, v_1v_4, v_4v_2\}$ does satisfy both the requests of the thesis.

If $|F| = n > 4$, let us suppose that the thesis holds for each 4-dense forest with $n-1$ vertices. Let $f$ be the vertex with the smallest degree in $F$. Such a vertex is a 4-leaf of $F$ hence, by Lemma 3.11, the graph $F' := F_{V(F)-\{f\}}$ is a 4-dense forest with $n-1$ vertices.

By the inductive hypothesis, there is a partition of $E(F')$ in two disjoint sets $A'_1$ and $A'_2$ such that points (i) and (ii) are satisfied. We have the following cases.

- $deg_F(f) \leq 2$. In order to prove the existence of a partition of the edges of $F$ in two sets which satisfy the condition (i) of the thesis, we have to distinguish the following cases.
  - $deg_F(f) = 0$. Obvious: the partition of the edges of $F$ in the following disjoint sets:
$$\begin{aligned} A_1 &:= A'_1 \\ A_2 &:= A'_2. \end{aligned}$$
    satisfies points (i) and (ii), by the hypotheses on $A'_1$ and $A'_2$.
  - $deg_F(f) = 1$. If we denote by $a$ the unique edge of $F$ which is incident in $f$, the partition of $E(F)$ in the two following disjoint sets satisfies the condition (i) of the thesis:
$$\begin{aligned} A_1 &:= A'_1 \cup \{a\} \\ A_2 &:= A'_2. \end{aligned}$$
  - $deg_F(f) = 2$. Let $v'$ and $v''$ be the vertices to which $f$ is adjacent. We can distinguish the following cases.
    * $v'v'' \notin E(F)$. Therefore the partition of $E(F)$ in the two following disjoint sets satisfies the condition (i) of the thesis:
$$\begin{aligned} A_1 &:= A'_1 \cup \{fv', fv''\} \\ A_2 &:= A'_2. \end{aligned}$$
    * $v'v'' \in E(F)$. Let us suppose that $v'v'' \in A'_1$ (the other case is analogous), the following partition of $E(F)$ in two disjoint sets satisfies the thesis:
$$\begin{aligned} A_1 &:= A'_1 \\ A_2 &:= A'_2 \cup \{fv', fv''\}. \end{aligned}$$

Finally, by the hypothesis on the sets $A_1'$ and $A_2'$, and since $F$ and $F'$ have the same 4-cliques, we can conclude that, for every partition of the edges of $F$ in two disjoint sets $A_1$ and $A_2$, if $A_1' \subset A_1$ and $A_2' \subset A_2$, such sets satisfy the condition (ii).

- $deg_F(f) = 3$. Let $C$ be the set of the maximal cliques of the subgraph of three vertices $F_{adj_F(f)}$. We have the following cases.
  - $|C| \geq 2$. To prove the existence of a partition of edges of $F$ in two disjoint sets satisfying the condition (i), we have to distinguish the following cases.
    * $|C| = 3$. Then $adj_F(f)$ is an independent set of $F$ hence, if we denote by $a_1, a_2, a_3$ the set of edges of $F$ incident in $F$, the following partition of $E(F)$ satisfies the condition (i):

$$\begin{aligned} A_1 &:= A_1' \cup \{a_1, a_2, a_3\} \\ A_2 &:= A_2'. \end{aligned}$$

    * $|C| = 2$. The graph $F_{adj_F(f)}$ has one or two edges. Let:

$$I := \{v', v''\}$$

      be an independent set of $F_{adj_F(f)}$ and let $v$ be the unique vertex in $adj_F(f) - I$. By the hypotheses of this case, there must be an edge $a$ between $v$ and a vertex of $I$. Let us suppose (the other case is analogous) that $a \in A_1'$, the partition of $E(F)$ in the two following disjoint sets satisfies the condition (i):

$$\begin{aligned} A_1 &:= A_1' \cup \{fv\} \\ A_2 &:= A_2' \cup \{fv', fv''\}. \end{aligned}$$

    Similarly to previous cases, by the hypothesis on the sets $A_1'$ and $A_2'$, and since $F$ and $F'$ have the same 4-cliques (since $|C| \geq 2$), we can conclude that, for each partition of the edges of $F$ in two disjoint sets $A_1$ e $A_2$, if $A_1' \subset A_1$ and $A_2' \subset A_2$ then such sets satisfy the condition (ii).
  - $|C| = 1$. $F_{adj_F(f)} \simeq K^3$ hence, by the inductive hypothesis, we know that the three edges of $F_{adj_F(f)}$ do not belong to only one set of the partition $\{A_1', A_2'\}$ of the edges of $F'$. Let us suppose (the other case is analogous) that $a := v'v''$ is the unique edge of $F_{adj_F(f)}$ included in $A_1'$. Let $v$ be the vertex in $adj_F(f) - \{v', v''\}$, the partition of $E(F)$ in the two following disjoint sets:

$$\begin{aligned} A_1 &:= A_1' \cup \{fv', fv\} \\ A_2 &:= A_2' \cup \{fv''\} \end{aligned}$$

    satisfies both the conditions of the thesis, by the hypotheses on the sets $A_1'$ and $A_2'$, and because the unique 4-clique of $F$ which does not belong to $F'$, is $\{v, v', v'', f\}$.

- $deg_F(f) = 4$. We can distinguish the following cases.
  - $\omega\left(F_{adj_F(f)}\right) \leq 2$, that is there are not in $F_{adj_F(f)}$ cliques of order greater than two. Hence, since $F_{adj_F(f)}$ is a graph of four vertices, we have two cases: $F_{adj_F(f)}$ is a cycle or $F_{adj_F(f)}$ is an acyclic graph (or a forest). Therefore, in both cases, we know that $F_{adj_F(f)}$ is 2-colourable.
    Let $c : adj_F(f) \to \{1, 2\}$ be a 2-colouring for $F_{adj_F(f)}$, the partition in the two following disjoint sets satisfies the condition (i):

$$A_1 := A_1' \cup \{fv \mid v \in adj_F(f) \wedge c(v) = 1\}$$

$$A_2 := A_2' \cup \{fv \mid v \in adj_F(f) \wedge c(v) = 2\}.$$

Let us suppose that there is a subgraph $S$ of $(V(F), A_1)$ isomorphic to $K^3$ (the other case is analogous). By the hypotheses on the sets $A'_1$ and $A'_2$, it is obvious that one of the vertices of $S$ must be $f$. We have that both the two edges of $S$ incident in $f$ belong to $A_1$; by the definition of $A_1$, for the two remaining vertices $v'$ and $v''$ of $S$ (obviously included in $adj_F(f)$ and adjacent in their turn) the following must hold:

$$c(v') = c(v'') = 1$$

against the hypothesis that $c$ is a 2-colouring for $F_{adj_F(f)}$.
Finally, by the hypotheses on the sets $A'_1$ and $A'_2$, and since $F$ and $F'$ have the same 4-cliques (by the hypothesis that $\omega\left(F_{adj_F(f)}\right) \leq 2$), we can conclude that the partition of the edges of $F$ in the two sets $A_1$ and $A_2$ satisfies the condition (ii).

- $\omega\left(F_{adj_F(f)}\right) = 3$, that is the graph $F_{adj_F(f)}$ has at least one 3-clique. We have to distinguish the following cases.
  * $F_{adj_F(f)}$ has *only one* 3-clique. Let $D = \{v_1, v_2, v_3\}$ be such a 3-clique, moreover let $v$ be the vertex of $F_{adj_F(f)}$ not included in $D$.
    By the inductive hypothesis, we know that the three edges of $F'_D$ do not belong to the same set of the partition $\{A'_1, A'_2\}$ of the edges of $F'$. Let us suppose (the other cases are anologous) that $a := v_1 v_2$ is the unique edge of $F'_D$ which is included in $A'_1$. The partition of the edges of the 4-dense forest:

    $$F'' := (V(F), E(F) - \{fv\})$$

    in the two disjoint sets:

    $$B_1 := A'_1 \cup \{fv_1, fv_3\}$$

    $$B_2 := A'_2 \cup \{fv_2\}$$

    satisfies both the conditions of the thesis, because of the hypotheses on the sets $A'_1$ and $A'_2$, and since the unique 4-clique of $F''$ which is not contained also in $F'$ is $\{v_1, v_2, v_3, f\}$.
    Let us suppose that there is an edge between $v$ and one vertex $u \in D$. Moreover, let us suppose that the edge $uv$ belongs to $B_1$ (once again, the other case is analogous). By this assumptions, the partition of the edges of $F$ in the sets $A_1 := B_1$ and $A_2 := B_2 \cup \{fv\}$ satisfies both the condition of the thesis (see Figure 9.1 $(i)$).
    Obviously, such a partition satisfies the thesis also if there is no edge between $v$ and a vertex in $D$.
  * $F_{adj_F(f)}$ has *two* 3-cliques. Therefore $F_{adj_F(f)}$ has five edges and one of them is shared by such 3-cliques. Let $v'$ and $v''$ be the vertices of $F_{adj_F(f)}$ which have the degree equal to 3 (in this hypothesis the edge shared by the 3-cliques is $v'v''$). Moreover, let $u'$ and $u''$ be the vertices with the degree equal to 2. We can distinguish the following cases.
    - The four shared edges $(v'u', v'u'', v''u', v''u'')$ are all included in one of the sets $A'_1$ and $A'_2$. Let us suppose that such a set is $A'_1$ (the other case is analogous). The partition of the edges of $F$ in the following sets:

      $$\begin{aligned} A_1 &:= A'_1 \cup \{fv'\} \\ A_2 &:= A'_2 \cup \{fv'', fu', fu''\} \end{aligned}$$

      satisfies both the conditions of the thesis (see Figure 9.1 $(ii)$).
    - All the (three) edges incident in one of the vertices $v'$ and $v''$, are contained in one of the sets $A'_1$ and $A'_2$. Let us assume that such vertex is $v'$ and that the set which contains all the edges (of $F_{adj_F(f)}$) which are incident in $v'$ is $A'_1$ (the other cases are analogous). Therefore, by

the hypotheses on $A'_1$ and $A'_2$, it is obvious that the other two edges of $F_{adj_F(f)}$ are contained in $A'_2$. In these hypotheses, the partition of the edges of $F$ in the following sets:

$$\begin{aligned} A_1 &:= A'_1 \cup \{fv''\} \\ A_2 &:= A'_2 \cup \{fv', fu', fu''\} \end{aligned}$$

satisfies both the conditions of the thesis (see Figure 9.1 $(iii)$).

- One of the two sets of edges $\{v'u', v''u''\}$ and $\{v''u', v'u''\}$ is included in the set ($A'_1$ or $A'_2$) to which $v'v''$ belongs. Let us suppose that $v'v''$ belongs to $A'_1$ and that:

$$\{v'u', v''u''\} \subseteq A'_1$$

(as always, the other cases are analogous). In this hypotheses, the partition of the edges of $F$ in the following sets:

$$\begin{aligned} A_1 &:= A'_1 \cup \{fu', fu''\} \\ A_2 &:= A'_2 \cup \{fv', fv''\} \end{aligned}$$

satisfies both the conditions of the thesis (see Figure 9.1 $(iv)$).

– $\omega\left(F_{adj_F(f)}\right) = 4$. Therefore $F_{adj_F(f)} \simeq K^4$ and hence, by the hypotheses on $A'_1$ and $A'_2$, the graphs:

$$\begin{aligned} P_1 &:= \left(adj_F(f), E\left(F_{adj_F(f)}\right) \cap A'_1\right) \\ P_2 &:= \left(adj_F(f), E\left(F_{adj_F(f)}\right) \cap A'_2\right) \end{aligned}$$

are paths of four vertices. Let $v_1$, $v_2$, $v_3$ and $v_4$ be the vertices of $F_{adj_F(f)}$ and let us suppose, without loss of generality, that the followings hold:

$$\begin{aligned} P_1 &= v_1v_2v_3v_4 \\ P_2 &= v_3v_1v_4v_2 \end{aligned}$$

(and so that $\{v_1v_2, v_2v_3, v_3v_4\} \subseteq A'_1$ and that $\{v_3v_1, v_1v_4, v_4v_2\} \subseteq A'_2$). In these hypotheses, the partition of the edges of $F$ in the two following sets:

$$\begin{aligned} A_1 &:= A'_1 \cup \{fv_1, fv_4\} \\ A_2 &:= A'_2 \cup \{fv_3, fv_2\} \end{aligned}$$

satisfies both the conditions of the thesis (see Figure 9.1 $(v)$). □

Therefore, by Proposition 9.2 and Observation 3.6, we have the following result.

**Theorem 9.3**
*For each integer $k \leq 4$, the problem $MT$ restricted to $k$-dense forests can be solved in constant time.*

From the proof of Proposition 9.2, we can easily draw an algorithm that, applied to a 4-dense forest and to a $k$-elimination for it, returns a partition of the edges of the graph which satisfies the requests of Problem 9.1. By Observation 3.6, it is obvious that such algorithm can be applied also to 2-dense forest and to 3-dense forest.

Because of the great deal of cases that such algorithm must consider, we omit an esplicit description of it.

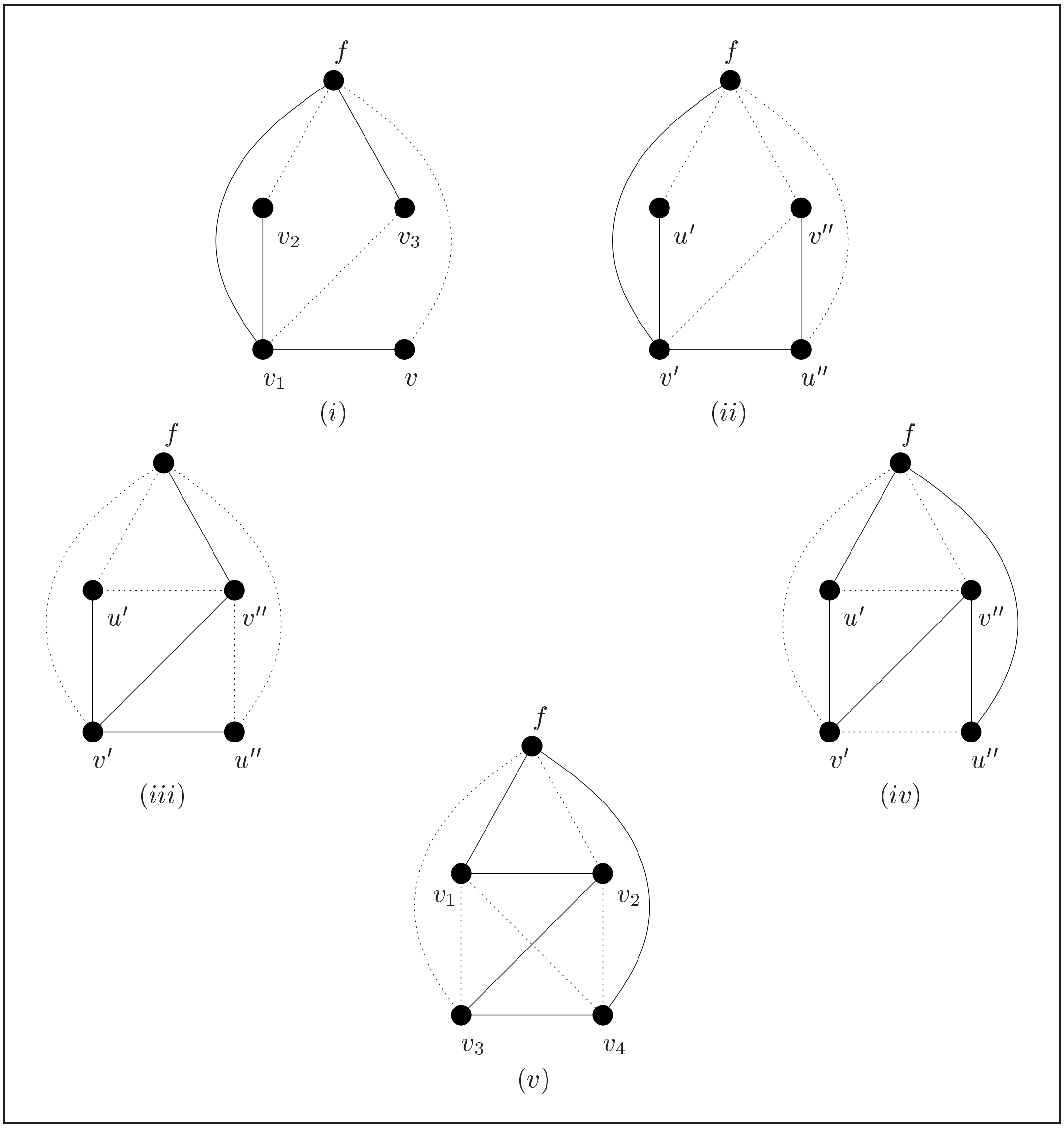


**Figure 9.1:** Some partition of the edges of the graph $F_{\{f\}\cup adj_F(f)}$ which are used in the proof of Proposition 9.2: the dotted edges belongs to $A_2$.

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