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Rank structure of generalized inverses of rectangular banded matrices*

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Abstract

We describe rank structures in generalized inverses of possibly rectangular banded matrices. In particular, we show that various kind of generalized inverses of rectangular banded matrices have submatrices whose rank depends on the bandwidth and on the nullity of the matrix. Moreover, we give an explicit representation formula for some generalized inverses of strictly banded matrices.

1 Introduction

Generalized inverses have important applications in matrix theory and in fields such as statistics, probability, linear systems theory [4, 11]. In particular, a special kind of generalized inverse, named after Moore and Penrose, allows to express in a useful way the solution in the least square sense of an overdetermined system of linear equations.

In this paper, we study in detail the structures of some generalized inverses of rectangular banded matrices. The results we are going to show generalize those in a series of papers by different authors [1, 3, 5, 12, 13], which address analogous structures appearing in inverses of square nonsingular banded matrices. A nice review on the literature about rank properties of this kind and their significance for square nonsingular matrices can be found in [14]. In the recent paper [2], by extending some results in [1], the structure of generalized inverses of square singular banded matrices is investigated. Here, further generalizations to rectangular banded matrices are shown. In particular, we consider various kinds of generalized inverses of a rectangular banded matrix A. We show that the submatrices belonging to a specified part of the inverses have rank that can be bounded by the sum of the bandwidth and the nullity of A^*. In the last section of this paper, we propose an explicit representation formula for the Moore-Penrose inverse of a rectangular banded matrix having non vanishing elements in the outermost diagonal.

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It is worth to mention that results of the same flavour have been obtained for matrices with displacement structure, such as Toeplitz, Hankel, Bezout, Cauchy, Vandermonde and their generalizations. Heinig and Hellinger in [7, 8, 9] show that these matrices possess generalized inverses with some kind of displacement structure. In particular, in [7, 8] representation formulas for Moore-Penrose inverses of Hankel and Toeplitz matrices are obtained.

The paper is organized as follows. In Section 2 we recall some definitions and basic results about banded matrices and generalized inverses. In Section 3 we prove our main theoretical results about the rank of the submatrices belonging to certain portions of generalized inverses of banded rectangular matrices. Section 4 contains the representation formula previously mentioned.

2 Preliminaries

Let $A \in \mathbb{C}^{m\times n}$, we denote with $\mathrm{Im}\,(A)$ the linear space spanned by the columns of A, and with $\mathrm{rank}(A)$ the dimension of $\mathrm{Im}\,(A)$. We denote with $\mathrm{Ker}\,(A)$ the nullspace of A and with $\mathrm{null}(A)$ the dimension of $\mathrm{Ker}\,(A)$. As well known $\mathrm{rank}(A) + \mathrm{null}(A) = n$. If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times r}$ we define

$$\begin{aligned} \mathrm{rank}_B(A) &= \mathrm{rank}(AB), \\ \mathrm{null}_B(A) &= \dim\left(\mathrm{Im}\,(B) \cap \mathrm{Ker}\,(A)\right). \end{aligned}$$

It is simple to prove that

$$\mathrm{rank}_B(A) + \mathrm{null}_B(A) = \mathrm{rank}(B). \tag{1}$$

Setting $C = AB$ we can rewrite the preceding relation as

$$\mathrm{rank}(C) = \mathrm{rank}(B) - \mathrm{null}_B(A). \tag{2}$$

Two subspaces $\mathcal{L}$ and $\mathcal{M}$ of $\mathbb{C}^n$ are said to be *complementary* if $\mathcal{L} \cap \mathcal{M} = \{0\}$ and $\mathcal{L} + \mathcal{M} = \mathbb{C}^n$. Thus, $\mathcal{L}$ and $\mathcal{M}$ are complementary if and only if every element of $\mathbb{C}^n$ can be expressed in a unique way as the sum of an element of $\mathcal{L}$ and an element of $\mathcal{M}$.

A matrix $P \in \mathbb{C}^{n\times n}$ is called a *projector* if $P^2 = P$. It can be shown, see [4], that if P is a projector then there are two complementary subspaces $\mathcal{L}$ and $\mathcal{M}$ of $\mathbb{C}^n$ such that $\mathrm{Im}\,(P) = \mathcal{L}$ and $\mathrm{Ker}\,(P) = \mathcal{M}$. We express this by writing $P = P_{\mathcal{L},\mathcal{M}}$ and by saying that P is a projector onto $\mathcal{L}$ along $\mathcal{M}$. Two projectors $P = P_{\mathcal{L},\mathcal{M}}$ and $Q = Q_{\mathcal{M},\mathcal{L}}$ are said to be *complementary projectors*. Clearly in this case we have

$$P + Q = I_n, \qquad PQ = QP = O_n,$$

where I_n and O_n are the $n\times n$ identity and null matrix, respectively. A projector $P = P_{\mathcal{L},\mathcal{M}}$ is defined to be *orthogonal* if the complementary subspaces $\mathcal{L}$ and $\mathcal{M}$ are orthogonal with respect to the usual Euclidean scalar product. It can be

easily shown that a projector P is orthogonal if an only if $P^* = P$, i.e. if and only if P is Hermitian.

Finally, for any matrix $A \equiv (a_{ij}) \in \mathbb{C}^{m\times n}$ we denote by $A(i_1 : i_2, j_1 : j_2)$ the submatrix consisting of all entries a_{ij} having $i_1 \le i \le i_2$ and $j_1 \le j \le j_2$.

2.1 Banded matrices

The following definition of banded matrix is basically the extension of the definition in [1] to the case where the matrix is rectangular.

Definition 1 *Let $A \equiv (a_{ij}) \in \mathbb{C}^{m\times n}$. The matrix A is called* lower k-banded *if $a_{ij} = 0$ for $j - i > k$ and it is called* strictly lower k-banded *if in addition $a_{ij} \neq 0$ for $j - i = k$. Analogously it is called* upper k-banded *if $a_{ij} = 0$ for $i - j < k$ and* strictly upper k-banded *if in addition $a_{ij} \neq 0$ for $i - j = k$.*

Note that a lower k-banded matrix with $k \ge n - 1$ is a full matrix, while with $k \le -m$ is the null matrix.

As a term of comparison with the results that we are going to show, it is worth to recall a couple of definitions and results that hold for square matrices.

Definition 2 *Let k be an integer such that $0 \le k \le n-1$. A matrix $A \in \mathbb{C}^{n\times n}$ is called* upper k-semiseparable *if*

$$\operatorname{rank}(A(1 : i + k, i + 1 : n)) \le k, \qquad i = 1, \dots, n - k - 1.$$

Analogously, A is called lower k-semiseparable *if*

$$\operatorname{rank}(A(i + 1 : n, 1 : i + k)) \le k, \qquad i = 1, \dots, n - k - 1.$$

Definition 3 *Let $A \in \mathbb{C}^{n\times n}$ be an upper k-semiseparable matrix. Then A is called* generator representable *if there exist two $n \times k$ matrices X, Y, called* generators*, and a lower $(-k)$-banded matrix L such that $A = XY^* + L$. Analogously, for a generator representable lower k-semiseparable matrix A we have $A = XY^* + U$ where U is an upper $(-k)$-banded matrix.*

Not all semiseparable matrices are also generator representable. If a nonsingular semiseparable matrix A is generator representable then all the inequalities in Definition 2 becomes equalities. Moreover, it has been proved in [15] that the set of generator representable matrices is dense in the set of semiseparable matrices. The importance of semiseparable matrices stems from the following result, see for example [1, 13, 14].

Theorem 1 *A nonsingular matrix A is (lower or upper) k-banded (with $k \ge 0$) if and only if its inverse is (upper or lower, respectively) k-semiseparable. Moreover, A is strictly k-banded if and only if its inverse is generator representable.*

2.2 Generalized inverses

Let $A \in \mathbb{C}^{m\times n}$ be a given matrix. The following matrix equations in the variable $X \in \mathbb{C}^{n\times m}$ are known as *Moore-Penrose equations*:

1. $AXA = A$
2. $XAX = X$
3. $(AX)^* = AX$
4. $(XA)^* = XA$.

A matrix X is called a *$\mathcal{S}$-inverse of A* if it fulfills a subset $\mathcal{S} \subseteq \{1,2,3,4\}$ of the Moore-Penrose equations. It can be easily proved, see [4, 11], that the $\{1,2,3,4\}$-inverse of A exists and is unique. It is called the *Moore-Penrose inverse of A* and customarily denoted with A^+. Any $\{1,2\}$-inverse of A is called a *reflexive generalized inverse of A*. In fact, if X is a $\{1,2\}$-inverse of A then A is a $\{1,2\}$-inverse of X. In [6] it is shown that a symmetric 1-semiseparable matrix admits a reflexive generalized inverse that has a block tridiagonal structure. Moreover, many displacement structured matrices have a reflexive generalized inverse with a displacement structure corresponding to that of the original matrix, see [7, 9]. $\{1,3\}$-inverses are solution operators for least square (inconsistent) linear equations, while $\{1,4\}$-inverses allow to compute minimum norm solutions of (consistent) linear equations, see [11, Chap. 3]. It is possible to prove that some of the above Moore-Penrose equations can be restated in terms of projectors.

Theorem 2 *Let $A \in \mathbb{C}^{m\times n}$ be a given matrix, and let $X \in \mathbb{C}^{n\times m}$ a $\mathcal{S}$-inverse of A where $\mathcal{S} \subseteq \{1,2,3,4\}$.*

(a) X is a $\{1\}$-inverse of A if and only if AX is a projector onto $\mathrm{Im}\,(A)$;

(b) X is a $\{2\}$-inverse of A if and only if XA is a projector onto $\mathrm{Im}\,(X)$;

(c) X is a $\{1,2\}$-inverse if and only if $AX = P_{\mathrm{Im}(A),\mathrm{Ker}(X)}$ and $XA = P_{\mathrm{Im}(X),\mathrm{Ker}(A)}$;

(d) X is a $\{1,3\}$-inverse if and only if $AX = P_{\mathrm{Im}(A),\mathrm{Ker}(A^*)}$;

(e) X is a $\{1,4\}$-inverse if and only if $XA = P_{\mathrm{Im}(A^*),\mathrm{Ker}(A)}$.

Proof 1 *(a) If X is a $\{1\}$-inverse of A we have $AXA = A$, hence $(AX)(AX) = (AX)^2 = AX$, from which it follows that AX is a projector. Moreover $\mathrm{Im}\,(A) = \mathrm{Im}\,(AXA) \subseteq \mathrm{Im}\,(AX) \subseteq \mathrm{Im}\,(A)$, which means that $\mathrm{Im}\,(AX) = \mathrm{Im}\,(A)$, hence AX is a projector onto $\mathrm{Im}\,(A)$. Conversely, if $P = AX$ is a projector onto $\mathrm{Im}\,(A)$, we have $PA = A$ and hence $AXA = A$. (b) is proved similarly to (a). (c) It remains to prove only that X if is a $\{1,2\}$-inverse of A then the projectors AX and XA are such that $\mathrm{Ker}\,(AX) = \mathrm{Ker}\,(X)$ and $\mathrm{Ker}\,(XA) = \mathrm{Ker}\,(A)$. Clearly, $\mathrm{Ker}\,(X) \subseteq \mathrm{Ker}\,(AX)$. On the other hand, if y is such that $AXy = 0$,*

then it holds $0 = AXy = XAXy = Xy$*, hence* $y \in \mathrm{Ker}\,(X)$*. Similarly for* XA *it can be proved that if* $AXA = A$*, then* $\mathrm{Ker}\,(XA) = \mathrm{Ker}\,(A)$*. (d) Since* X *is in particular a* $\{1\}$*-inverse of* A*, it remains to prove that* X *is also a* $\{3\}$*-inverse if and only if* $P = AX$ *projects along* $\mathrm{Ker}\,(A^*)$*. But, since* $\mathrm{Ker}\,(A^*) = \mathrm{Im}\,(A)^{\perp}$*,* AX *projects along* $\mathrm{Ker}\,(A^*)$ *if and only if it is orthogonal. And* AX *is orthogonal if and only if it is Hermitian, i.e. if and only if* $AX = (AX)^*$*. (e) is proved similarly to (d).* ■

Note that if $X = A^+$ is the Moore-Penrose inverse of A, then all the statements of Theorem 2 hold. As a consequence, one has $\mathrm{Ker}\,(A^*) = \mathrm{Ker}\,(A^+)$.

3 Generalized inverses of rectangular matrices

In this section we analyze the structure of generalized inverses of rectangular banded matrices. In particular we show that generalized inverses of a lower k banded matrix A have the upper-right submatrices whose rank does not exceed $k + \mathrm{null}\,(A^*)$. Rank properties of this kind have been emphasized by many authors for the inverses of nonsingular square banded matrices. An introduction to this subject can be found in the paper of Strang and Nguyen [14].

First of all let us recall a result due to E. Asplund [1]. Here we translate in matrix language the original formulation given in terms of linear operators.

Lemma 1 *Let* $R, S \in \mathbb{C}^{n\times n}$ *and* $T, U \in \mathbb{C}^{n\times n}$ *be two pairs of complementary projectors and let* $A \in \mathbb{C}^{n\times n}$ *be nonsingular. The equation* $TAS = O_n$ *holds true if and only if*

$$\mathrm{rank}(RA^{-1}U) \leq \mathrm{rank}(R) - \mathrm{rank}(T).$$

Note that, in the above mentioned result, one has $\mathrm{rank}(R) - \mathrm{rank}(T) = \mathrm{rank}(U) - \mathrm{rank}(S)$. The following lemma extends Asplund's result to various kinds of generalized inverses, also in the rectangular case.

Lemma 2 *Let* $R, S \in \mathbb{C}^{n\times n}$ *and* $T, U \in \mathbb{C}^{m\times m}$ *be two pairwise complementary projectors and let* $A \in \mathbb{C}^{m\times n}$*. If* $TAS = O_{m\times n}$ *and* X *is either any* $\{1,2\}$*-inverse,* $\{1,3\}$*-inverse or* $\{1,4\}$*-inverse of* A*, then it holds*

$$\mathrm{rank}\,(RXU) \leq \mathrm{rank}\,(U) - \mathrm{rank}\,(S) + \mathrm{null}\,(A)\,.$$

Proof 2 *Let us consider the case where* X *is any* $\{1,2\}$*- or* $\{1,4\}$*-inverse of* A*. From* $T + U = I_m$ *and* $TAS = O$*, we have* $AS = (T+U)AS = UAS$*. Then, using* $U^2 = U$*, we have*

$$RXAS = RXUAS = (RXU)(UAS).$$

Let C *be a matrix having* m *rows and* $c = \mathrm{rank}\,(U) - \mathrm{rank}\,(UAS)$ *linearly independent columns, such that the matrix* $W = [UAS|C]$ *span the same subspace spanned by* U*, that is*

$$\mathrm{Im}\,(W) = \mathrm{Im}\,([UAS|C]) = \mathrm{Im}\,(U)\,. \tag{3}$$

Obviously, $\operatorname{rank}(W) = \operatorname{rank}(U)$. *Since the columns of* C *are linearly independent, we have* $\operatorname{rank}(C) = c = \operatorname{rank}(U) - \operatorname{rank}(UAS)$.

From (3), we have $\operatorname{rank}(RXU) = \operatorname{rank}(RXUW)$, *and since*

$$\begin{aligned}\operatorname{rank}(RXUW) &\leq \operatorname{rank}\left(RXU^2AS\right) + \operatorname{rank}(RXUC)\\ &= \operatorname{rank}(RXAS) + \operatorname{rank}(RXUC)\\ &\leq \operatorname{rank}(RXAS) + \operatorname{rank}(C),\end{aligned}$$

we find

$$\operatorname{rank}(RXU) \leq \operatorname{rank}(RXAS) + \operatorname{rank}(U) - \operatorname{rank}(UAS). \tag{4}$$

If X *is either a* $\{1,2\}$*- or* $\{1,4\}$*-inverse of* A *then, from claims (c) and (e) of Theorem 2,* XA *is a projector along* $\operatorname{Ker}(A)$. *Let* $P = XA$ *be such a projector. Note that if* X *is a* $\{1,2\}$*-inverse, then* P *is a projector onto* $\operatorname{Im}(X)$, *while if* X *is a* $\{1,4\}$*-inverse,* P *projects onto* $\operatorname{Im}(A^*)$. *However, in both cases we have* $\operatorname{Ker}(A) = \operatorname{Ker}(P)$.

Let us consider the matrix $RXAS = RPS$. *By applying twice equation (2) we obtain*

$$\begin{aligned}\operatorname{rank}(RPS) &= \operatorname{rank}(PS) - \operatorname{null}_{PS}(R)\\ &= \operatorname{rank}(S) - \operatorname{null}_S(P) - \operatorname{null}_{PS}(R).\end{aligned} \tag{5}$$

Furthermore, using again (2) we obtain

$$\begin{aligned}\operatorname{null}_S(P) &= \dim(\operatorname{Im}(S) \cap \operatorname{Ker}(P))\\ &= \dim(\operatorname{Im}(S) \cap \operatorname{Ker}(A)) = \operatorname{null}_S(A)\\ &= \operatorname{rank}(S) - \operatorname{rank}(AS) = \operatorname{rank}(S) - \operatorname{rank}(UAS).\end{aligned} \tag{6}$$

Let us consider the quantity $\operatorname{null}_{PS}(R)$. *By definition, we have*

$$\operatorname{null}_{PS}(R) = \dim(\operatorname{Im}(PS) \cap \operatorname{Ker}(R)). \tag{7}$$

Since R *and* S *are complementary projectors we have*

$$\operatorname{Ker}(R) = \operatorname{Im}(S). \tag{8}$$

It can be proved that $\operatorname{Im}(PS) \cap \operatorname{Im}(S) = \operatorname{Im}(P) \cap \operatorname{Im}(S)$. *In fact,* $\operatorname{Im}(PS) \subseteq \operatorname{Im}(P)$, *hence* $\operatorname{Im}(PS) \cap \operatorname{Im}(S) \subseteq \operatorname{Im}(P) \cap \operatorname{Im}(S)$. *Moreover,* $\operatorname{Im}(S) \cap \operatorname{Im}(P) \subseteq \operatorname{Im}(PS) \cap \operatorname{Im}(S)$, *since* $y \in \operatorname{Im}(P)$ *implies* $Py = y$, *but if* $y \in \operatorname{Im}(S)$, *there exists* z *such that* $Sz = y$. *Hence,* $PSz = y$, *which guarantees that* $y \in \operatorname{Im}(PS)$. *From (7) and (8) it follows*

$$\operatorname{null}_{PS}(R) = \dim(\operatorname{Im}(P) \cap \operatorname{Im}(S)) \geq \operatorname{rank}(S) + \operatorname{rank}(P) - n. \tag{9}$$

The equality $\operatorname{Ker}(P) = \operatorname{Ker}(A)$ *implies* $\operatorname{rank}(P) = n - \operatorname{null}(A)$. *Putting together (5), (6) and (9) we get*

$$\operatorname{rank}(RPS) \leq \operatorname{rank}(UAS) - \operatorname{rank}(S) + \operatorname{null}(A). \tag{10}$$

Finally, from (4) and (10), we have

$$\operatorname{rank}(RXU) \leq \operatorname{rank}(U) - \operatorname{rank}(S) + \operatorname{null}(A).$$

If X is a $\{1,3\}$-inverse of A, then X^* is a $\{1,4\}$-inverse of A^*. From the equation $S^*A^*T^* = O_{n\times m}$ and the foregoing result, we have

$$\begin{aligned}\operatorname{rank}(RXU) &= \operatorname{rank}(U^*X^*R^*)\\ &\leq \operatorname{rank}(R^*) - \operatorname{rank}(T^*) + \operatorname{null}(A^*)\\ &= \operatorname{rank}(R) - \operatorname{rank}(T) + \operatorname{null}(A^*).\end{aligned}$$

Since $\operatorname{null}(A^*) = m - n + \operatorname{null}(A)$*, and* $\operatorname{rank}(R) - \operatorname{rank}(T) = n - \operatorname{rank}(S) - m + \operatorname{rank}(U)$*, we obtain the claimed inequality for the rank of the matrix RXU.* ■

The previous lemma allows to prove the central result of this paper regarding the low rank structure of submatrices of generalized inverses of banded matrices.

Theorem 3 *Let $A \in \mathbb{C}^{m\times n}$ be a lower k-banded matrix, and let $X \in \mathbb{C}^{n\times m}$ be either a $\{1,2\}$-inverse, a $\{1,3\}$-inverse, or a $\{1,4\}$-inverse of A. Then, for $t = 1,\ldots,n-k$,*

$$\operatorname{rank}(X(1:t+k, t+1:m)) \leq k + \operatorname{null}(A^*).$$

Proof 3 *Suppose that $X \equiv (x_{ij})$ is a $\{1,h\}$-inverse of A, where $h \in \{2,3,4\}$. Let $C = X(1:t+k, t+1:m)$ for some $t = 1,\ldots,n-k$. Consider the following 2×2 block diagonal matrices (off-diagonal blocks are zeros):*

$$R = \begin{pmatrix} I_r & \\ & O_{n-r}\end{pmatrix}, \quad U = \begin{pmatrix} O_t & \\ & I_{m-t}\end{pmatrix},$$

where $r = t+k$. Moreover, let $S = I_n - R$ and $T = I_m - U$. Then C is also a submatrix of RXU, and R, S, T, U fulfill the hypotheses of Lemma 2. We have $TAS = O_{m\times n}$, because of the banded structure of A. Then by Lemma 2 it follows:

$$\begin{aligned}\operatorname{rank}(C) &\leq \operatorname{rank}(RXU)\\ &\leq \operatorname{rank}(U) - \operatorname{rank}(S) + \operatorname{null}(A)\\ &= m - t - (n-r) + \operatorname{null}(A)\\ &= m - n + k + \operatorname{null}(A) = k + \operatorname{null}(A^*).\end{aligned}$$

■

The bound in the above theorem can be reached even in the square case.

Example 1 *Let us consider the following tridiagonal matrix*

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix T is singular and $\mathrm{null}\,(T^*) = \mathrm{null}\,(T) = 1$, *moreover T is a lower (and upper)* 1*-banded matrix. The Moore-Penrose inverse of T is*

$$T^+ = \frac{1}{3}\begin{pmatrix} 0 & 2 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 \end{pmatrix}.$$

From Theorem 3 we have that the rank of every submatrix of T^+ above (or below) the main diagonal has rank less or equal to 2. In fact, in this example every submatrix above the diagonal with at least two columns and two rows has rank equal to 2. ■

The following example shows another "extreme" situation.

Example 2 *The matrix $Z \equiv (z_{ij})$, such that $z_{ij} = 0$ if $j \neq i+1$ and $z_{i,i+1} = 1$ has $k = 1$ and* $\mathrm{null}\,(Z^*) = 1$. *However, $Z^+ = Z^*$, and all the ranks of the submatrices involved in Theorem 3 are equal to zero.* ■

We end this section by presenting one more example that, in some sense, generalizes Example 2. We need the following result from [11, Lemma 2.5.2]:

Lemma 3 *Let $A = MN$ be a rank factorization of the matrix $A \in \mathbb{C}^{m\times n}$, that is, $M \in \mathbb{C}^{m\times\rho}$ and $N \in \mathbb{C}^{\rho\times n}$ where $\rho = \mathrm{rank}\,(A)$. Then $X \in \mathbb{C}^{n\times m}$ is a $\{1,2\}$-inverse of A if and only if $X = PQ$, where $P \in \mathbb{C}^{n\times\rho}$ and $Q \in \mathbb{C}^{\rho\times m}$ are respectively a right inverse of N and a left inverse of M, that is, $NP = QM = I_\rho$.*

Example 3 *Let $A \in \mathbb{C}^{m\times n}$ be a strictly lower k-banded matrix. Suppose that $k \geq \max\{0, n-m\}$ so that A has no leading zero rows nor trailing zero columns. In this setting the last $n-k$ columns of A are linearly independent, because of their lower triangular structure. As a consequence, if A has linearly dependent columns, these columns can be chosen within the first k. Let $\nu = \mathrm{null}\,(A) = n - \mathrm{rank}\,(A) = n - \rho$. Without loss of generality, we may suppose that the first ν columns are linearly dependent from the others.*

Now let

$$M = A(1:m, \nu+1:n), \qquad N = [N_1 \,|\, I_\rho],$$

where N_1 is such that $MN_1 = A(1:m, 1:\nu)$. Hence $A = MN$ is a rank factorization of A. We observe that M is a strictly lower $(k-\nu)$-banded matrix. By adding $m-\rho$ suitably chosen columns, we can embed M into a square nonsingular matrix of order m, say $\hat{M} = [M \mid Z]$, that is still strictly lower $(k-\nu)$-banded. By virtue of Theorem 1, the matrix $\hat{M}^{-1}$ is upper $(k-\nu)$-semiseparable and generator representable. This guarantees

$$\operatorname{rank}\left(\hat{M}^{-1}(1:i+k-\nu, i+1:m)\right) = k-\nu,$$

for $i = 1, \ldots, n-k+\nu-1$. From the equation $\hat{M}^{-1}\hat{M} = I_m$, we see that the first ρ rows of $\hat{M}^{-1}$ are a left inverse of M. Let $Q = \hat{M}^{-1}(1:\rho, 1:m)$. Moreover, the matrix $P = [O_{\rho\times\nu} \mid I_\rho]^T$ is a right inverse of N. From Lemma 3 we have that the matrix $X = PQ$ is a $\{1,2\}$-inverse of A. We see that

$$X = \begin{pmatrix} O_{\nu\times\rho} \\ I_\rho \end{pmatrix} Q = \begin{pmatrix} O_{\nu\times m} \\ Q \end{pmatrix}.$$

Moreover, for $i = 1, \ldots, n-k$ we obtain

$$\begin{aligned} &\operatorname{rank}\left(X(1:i+k, i+1:m)\right) = \\ &\quad = \operatorname{rank}\left(Q(1:i+k-\nu, i+1:m)\right) = k-\nu. \end{aligned} \tag{11}$$

■

We conjecture that, among all $\{1,2\}$-inverses of a strictly lower k-banded matrix A, the matrix X constructed in Example 3 has the smallest ranks in the submatrices considered in equation (11). Remark that the Moore-Penrose inverse of the matrix Z in Example 2 fulfills exactly equation (11).

4 Explicit representation of Moore-Penrose inverse

Many formulas have been found for generalized inverses of a block partitioned matrix assuming that one of the blocks is nonsingular. The books [4, 11] make a review of these formulas and indicate many useful references. In this section we propose just another formula of this family but tailored on banded matrices. Our goal is to generalize the simple representation formula presented in [1] for square strictly banded nonsingular matrices.

Let $A \in \mathbb{C}^{m\times n}$ be a strictly lower k-banded matrix. In this section we assume $\max\{0, n-m\} \le k \le n-1$. The considered matrix A can be rewritten in terms of blocks in the form

$$A = \begin{pmatrix} A_{11} & L \\ A_{21} & A_{22} \end{pmatrix}$$

where $L \in \mathbb{C}^{(n-k)\times(n-k)}$ is nonsingular lower triangular.

As well known we can factorize A as follows

$$A = \begin{pmatrix} I_{n-k} & O \\ Q & I_{m-n+k} \end{pmatrix} \begin{pmatrix} O & L \\ S & O \end{pmatrix} \begin{pmatrix} I_k & O \\ P & I_{n-k} \end{pmatrix}, \tag{12}$$

where $Q = A_{22}L^{-1} \in \mathbb{C}^{(m-n+k)\times(n-k)}$, $P = L^{-1}A_{11} \in \mathbb{C}^{(n-k)\times k}$ and where $S = A_{21} - A_{22}L^{-1}A_{11} \in \mathbb{C}^{(m-n+k)\times k}$ is known as Schur complement of L in A. Let $\mathrm{rank}(S) = r$, with $0 \le r \le \min\{k,\, m-n+k\}$ and let $S = S_1S_2$ where $S_1 \in \mathbb{C}^{(m-n+k)\times r}$ and $S_2 \in \mathbb{C}^{r\times k}$ are two matrices whose rank is r (for the case where $r = 0$ we need only to assume that the product of a $m \times 0$ matrix by a $0 \times n$ matrix is the $m \times n$ mull matrix). Then

$$A = \begin{pmatrix} I_{n-k} & O \\ Q & S_1 \end{pmatrix} \begin{pmatrix} O & L \\ I_r & O \end{pmatrix} \begin{pmatrix} S_2 & O \\ P & I_{n-k} \end{pmatrix}. \tag{13}$$

We notice that $\mathrm{rank}(A) = n-k+r$ so that $\mathrm{null}(A) = k-r$. This factorization allows to compute A^+ by simply computing the product of the Moore-Penrose inverses of the factors in the reverse order. Hence

$$A^+ = \begin{pmatrix} S_2 & O \\ P & I_{n-k} \end{pmatrix}^+ \begin{pmatrix} O & I_r \\ L^{-1} & O \end{pmatrix} \begin{pmatrix} I_{n-k} & O \\ Q & S_1 \end{pmatrix}^+. \tag{14}$$

The two block triangular pseudo-inverse matrices appearing in the previous equations have full rank, hence

$$\begin{pmatrix} S_2 & O \\ P & I_{n-k} \end{pmatrix}^+ = \begin{pmatrix} S_2 & O \\ P & I_{n-k} \end{pmatrix}^* \begin{pmatrix} S_2S_2^* & S_2P^* \\ PS_2^* & PP^* + I_{n-k} \end{pmatrix}^{-1},$$

and

$$\begin{pmatrix} I_{n-k} & O \\ Q & S_1 \end{pmatrix}^+ = \begin{pmatrix} I_{n-k} + Q^*Q & Q^*S_1 \\ S_1^*Q & S_1^*S_1 \end{pmatrix}^{-1} \begin{pmatrix} I_{n-k} & O \\ Q & S_1 \end{pmatrix}^*.$$

We get

$$\begin{aligned} A^+ \;=\; & \begin{pmatrix} S_2^* & P^* \\ O & I_{n-k} \end{pmatrix} \begin{pmatrix} S_2S_2^* & S_2P^* \\ PS_2^* & I_{n-k} + PP^* \end{pmatrix}^{-1} \\ & \begin{pmatrix} O & I_r \\ L^{-1} & O \end{pmatrix} \begin{pmatrix} I_{n-k} + Q^*Q & Q^*S_1 \\ S_1^*Q & S_1^*S_1 \end{pmatrix}^{-1} \begin{pmatrix} I_{n-k} & Q^* \\ O & S_1^* \end{pmatrix}. \end{aligned}$$

By Morrison formula we find $(I_{n-k} + PP^*)^{-1} = I_{n-k} - P\widetilde{P}P^*$ where $\widetilde{P} = (I_k + P^*P)^{-1}$. Analogously $(I_{n-k} + Q^*Q)^{-1} = I_{n-k} - Q^*\widetilde{Q}Q$ where $\widetilde{Q} = (I_{m-n+k} + QQ^*)^{-1}$. Observe that $P^*P\widetilde{P} = \widetilde{P}P^*P = I_k - \widetilde{P}$ and $\widetilde{Q}QQ^* = QQ^*\widetilde{Q} = I_{m-n+k} - \widetilde{Q}$. Using these relations we obtain

$$\begin{aligned} \begin{pmatrix} S_2S_2^* & S_2P^* \\ PS_2^* & I_{n-k} + PP^* \end{pmatrix}^{-1} \;=\; & \begin{pmatrix} O & O \\ O & I_{n-k} - P\widetilde{P}P^* \end{pmatrix} \\ & + \begin{pmatrix} I_r \\ -P\widetilde{P}S_2^* \end{pmatrix} (S_2\widetilde{P}S_2^*)^{-1} \begin{pmatrix} I_r & -S_2\widetilde{P}P^* \end{pmatrix}, \end{aligned}$$

and analogously

$$\begin{pmatrix} I_{n-k}+Q^*Q & Q^*S_1 \\ S_1^*Q & S_1^*S_1 \end{pmatrix}^{-1} = \begin{pmatrix} I_{n-k}-Q^*\widetilde{Q}Q & O \\ O & O \end{pmatrix} + \begin{pmatrix} -Q^*\widetilde{Q}S_1 \\ I_r \end{pmatrix} (S_1^*\widetilde{Q}S_1)^{-1} \begin{pmatrix} -S_1^*\widetilde{Q}Q & I_r \end{pmatrix}.$$

We observe that

$$\begin{pmatrix} S_2^* & P^* \\ O & I_{n-k} \end{pmatrix} \begin{pmatrix} S_2S_2^* & S_2P^* \\ PS_2^* & I_{n-k}+PP^* \end{pmatrix}^{-1} = \begin{pmatrix} O & O \\ O & I_{n-k} \end{pmatrix} + \begin{pmatrix} I_k \\ -P \end{pmatrix} \begin{pmatrix} \widetilde{P}S_2^*(S_2\widetilde{P}S_2^*)^{-1} & \widetilde{P}P^* - \widetilde{P}S_2^*(S_2\widetilde{P}S_2^*)^{-1}S_2\widetilde{P}P^* \end{pmatrix},$$

and analogously

$$\begin{pmatrix} I_{n-k}+Q^*Q & Q^*S_1 \\ S_1^*Q & S_1^*S_1 \end{pmatrix}^{-1} \begin{pmatrix} I_{n-k} & Q^* \\ O & S_1^* \end{pmatrix} = \begin{pmatrix} I_{n-k} & O \\ O & O \end{pmatrix} + \begin{pmatrix} Q^*\widetilde{Q} - Q^*\widetilde{Q}S_1(S_1^*\widetilde{Q}S_1)^{-1}S_1^*\widetilde{Q} \\ (S_1^*\widetilde{Q}S_1)^{-1}S_1^*\widetilde{Q} \end{pmatrix} \begin{pmatrix} -Q & I_{m-n+k} \end{pmatrix}.$$

Just for keeping formulas to a reasonable length let us set

$$\begin{aligned} \Pi_1 &= \widetilde{P}S_2^*(S_2\widetilde{P}S_2^*)^{-1}, \\ \Pi_2 &= \widetilde{P}P^* - \widetilde{P}S_2^*(S_2\widetilde{P}S_2^*)^{-1}S_2\widetilde{P}P^*, \\ \Theta_1 &= Q^*\widetilde{Q} - Q^*\widetilde{Q}S_1(S_1^*\widetilde{Q}S_1)^{-1}S_1^*\widetilde{Q}, \\ \Theta_2 &= (S_1^*\widetilde{Q}S_1)^{-1}S_1^*\widetilde{Q}. \end{aligned}$$

At this point we can write an explicit expression for A^+.

$$\begin{aligned} A^+ &= \begin{pmatrix} O & O \\ L^{-1} & O \end{pmatrix} + \begin{pmatrix} I_k \\ -P \end{pmatrix} (\Pi_2L^{-1}\Theta_1 + \Pi_1\Theta_2) \begin{pmatrix} -Q & I_{m-n+k} \end{pmatrix} \\ &\quad + \begin{pmatrix} I_k \\ -P \end{pmatrix} \Pi_2L^{-1} \begin{pmatrix} I_{n-k} & O \end{pmatrix} \\ &\quad + \begin{pmatrix} O \\ I_{n-k} \end{pmatrix} L^{-1}\Theta_1 \begin{pmatrix} -Q & I_{m-n+k} \end{pmatrix}. \end{aligned}$$

In the case where $S = O$, as happens for example for square singular irreducible Hessenberg matrices, the formula for A^+ becomes much more simpler.

$$\begin{aligned} A^+ &= \begin{pmatrix} O & O \\ L^{-1} & O \end{pmatrix} + \begin{pmatrix} I_k \\ -P \end{pmatrix} \widetilde{P}P^*L^{-1}Q^*\widetilde{Q} \begin{pmatrix} -Q & I_{m-n+k} \end{pmatrix} \\ &\quad + \begin{pmatrix} I_k \\ -P \end{pmatrix} \widetilde{P}P^*L^{-1} \begin{pmatrix} I_{n-k} & O \end{pmatrix} \\ &\quad + \begin{pmatrix} O \\ I_{n-k} \end{pmatrix} L^{-1}Q^*\widetilde{Q} \begin{pmatrix} -Q & I_{m-n+k} \end{pmatrix}. \end{aligned}$$

5 Conclusion and further work

In this paper we studied the structure of generalized inverses of rectangular banded matrices, showing that various kind of generalized inverses have submatrices whose rank is limited by a constant depending on the bandwidth and on the nullity of the banded matrix. However, the reverse is not true, since generalized inverses of singular semiseparable matrices are not necessarily banded. It would be interesting to investigate both the possibility of finding among the class of generalized inverses of a k-banded matrix a particular one with a k-semiseparable structure and among the generalized inverses of a semiseparable matrix a particular one with banded structure.

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