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Robust Portfolio Asset Allocation: models and algorithmic approaches

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Abstract: Many financial optimization problems involve future values of security prices, interest rates and exchange rates which are not known in advance, but can only be forecasted or estimated. Such problems fit perfectly into the framework of Robust Optimization that, given optimization problems with uncertain parameters, looks for solutions that will achieve good objective function values for the realization of these parameters in given uncertainty sets. In finance, Robust Optimization offers vehicles to incorporate the estimation of uncertain parameters into the decision making process. This is true, for example, in portfolio asset allocation.

Starting from the robust counterparts of the classical mean-variance portfolio problems, in this paper we review some mathematical models that have been recently proposed in the literature to address uncertainty in portfolio asset allocation problems. For some of these, we focus also on algorithmic approaches and computataional issues.

Finally, we analyze the relationship between robustness and risk measures.

1 Introduction

Portfolio selection problems were formulated for the first time by Markowitz in 1952. They consist in allocating capital over a number of available assets in order to maximize the 'return' on the investment while minimizing the 'risk' using mathematical techniques. In the proposed models, the return is measured by the expected value of the random portfolio return, while the risk is quantified by the variance of the portfolio (*mean-variance models*).

Despite the strong theoretical support, the availability of efficient computer codes to solve them and the elegance of the models, they present some practical pitfalls: the optimal portfolios are not well diversified; in fact they tend to concentrate on a small subset of the available securities and, above all, they are often very sensitive to changes in the input parameters.

Several techniques have been suggested to reduce the sensitivity of Markowitz

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models. One of them is represented by Robust Optimization. This framework refers to modeling of optimization problems with data uncertainty to obtain a solution that is guaranteed to be 'good' for all possible realizations of the uncertain parameters in given uncertainty sets. Uncertainty in the parameters is in fact described through *uncertainty sets* that contain possible values that may be realized for the uncertain parameters.
Recently, robust models together with related algorithmic approaches have been proposed in the literature to address uncertainty in portfolio asset allocation problems. Some of these models are described in [12], where an overview of robust models in asset allocation problems is proposed.
Aim of this paper is to enlarge the overview in [12], by introducing further models and synthetizing also some computational aspects. This is the subject of Section 2. Furthermore, in Section 3 we show a concept of robustness which is tied to the concept of measures of risk.

2 Robust asset allocation

2.1 The classical models

Optimal portfolio asset allocation problems can be formulated mathematically as quadratic programming (QP) problems [18], [26]. Specifically, some of them can be formulated as convex QP, that refers to minimizing a quadratic function subject to linear constraints.

Let n be the number of the available assets, and

$$X = \left\{ x \in \mathcal{R}^n | \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0, \imath = 1, ..n \right\} \tag{1}$$

be the set of the feasible portfolios.

Furthermore, let μ be the estimated expected return vector of the given assets, while matrix Q be the covariance matrix of these returns.

Then, the classical *mean-variance optimization (MVO) models* of Markowitz can be formulated as follows:

1) Maximize the expected return subject to an upper limit on the variance:

$$\begin{aligned} & \max \mu^T x \\ s.t. \quad & x^T Q x \leq \sigma \\ & x \in X; \end{aligned} \tag{2}$$

2) minimize the variance subject to a lower limit on the expected return:

$$
\begin{aligned}
&\min x^T Q x \\
&s.t. \quad \mu^T x \geq R \\
&x \in X;
\end{aligned}
\tag{3}
$$

3) maximize the *risk-adjusted expected return*:

$$
\max \mu^T x - \lambda x^T Q x \tag{4}
$$

$$
s.t. \quad x \in X
$$

where $\lambda \in \mathcal{R}$ denotes a risk-adversion parameter.
These three models are parametrized by the variance limit, the expected return limit and the risk-adversion parameter, respectively. Since the variance constraint is a nonlinear constraint, the first formulation can not be classified as a convex QP formulation, while the latter two are convex QP formulations.
A study of Black and Litterman [2] demonstrated that small changes in the expected returns, in particular, may have a substantial impact in the portfolio composition. It follows that, if the estimation errors in the expected returns are large, then they can significatly influence the optimal allocation. For practical applications, it is therefore crucial to incorporate the uncertainty about the accuracy of the estimates in the portfolio optimization process. Although mean-variance portfolio optimization seems to be less sensitive to inaccuracies in the estimate of the covariance matrix Q than to estimation errors in the expected returns, insurance against uncertainty in these estimates is recommended too, and it can be incorporated at not too large a cost.
A way to incorporate uncertainty in Markowitz models is to define suitable uncertainty sets for μ and Q, and to select the optimal portfolio with respect to the worst data realization according to the chosen uncertainty sets. The resulting models will be reviewed in the following section.

2.2 Robust MVO models

Assume that the uncertain mean return vector μ and the uncertain covariance matrix Q of the asset return belong to uncertain sets of the following form:

$$
U_\mu = \left\{\mu : \mu^L \leq \mu \leq \mu^U\right\} \text{ and } U_Q = \left\{Q : Q \succeq 0, Q^L \leq Q \leq Q^U\right\}.
$$

The end-points of the intervals may correspond to the extreme values of the corresponding statistic in historical data, in analyst estimates, in simulated scenarios. Alternatively, a modeler may choose a confidence level and then generate estimates of return and covariance parameters in the form of prediction intervals.

Based on the above introduced uncertainty sets, Koenig and Tütüncü [26] have formulated some robust counterparts of problem (4) and (3) by exploiting formulations previously introduced by Goldfarb and Iyengar [16] and by

Halldorsson and Tütüncü [17]. The first robust problem determines a feasible portfolio x such that its maximum risk-adjusted expected return, where both parameters vary in the given uncertainty sets, is the minimum ones among the feasible portfolios. On the other hand, the latter robust problem looks for a feasible portfolio which guarantees the lower limit R on the expected return also in the worst case, i.e., for the worst realization of parameter μ in U_μ, and which minimizes the variance in the worst realization of parameter Q according to the uncertainty set U_Q:

$$\max_{x \in X} \left\{ \min_{\mu \in U_\mu, Q \in U_Q} \mu^T x - \lambda x^T Q x \right\} \tag{5}$$

and

$$\begin{aligned} &\min \max_{Q \in U_Q} x^T Q x \\ &s.t. \ \min_{\mu \in U_\mu} \mu^T x \geq R, \\ &\quad x \in X \end{aligned} \tag{6}$$

Under certain simplifying assumptions, that is when Q^U is a positive semidefinite matrix, these robust problems can be reduced to pure MVO problems. In such a special case, the best asset allocation can in fact be determined by first fixing the worst-case input data in the considered uncertainty sets, that is μ^L for the uncertain mean return vector μ and Q^U for the uncertain covariance matrix Q, and then solving the resulting QP problems [26]. Without these assumptions, it is not possible to solve the robust asset allocation problems in a sequential manner. In the general case, the robust counterparts (5) and (6) can be solved using a nonlinear saddle-point formulation that involves semidefinite constraints [17].

An alternative method for modeling uncertainty was proposed by Goldfarb-Iyengar using the factor model [16]. Let us consider a standard factor model for representing returns, that is

$$r = \mu + V^T f + \varepsilon,$$

then the covariance matrix of the returns Q can be expressed as

$$Q = V^T F V + D$$

where V denotes the matrix of factor loading, F is the covariance matrix of the factor returns and D is the diagonal matrix of the error term variances.
The individual elements d_i of the covariance matrix D are assumed to lie in an interval $[d_i, \bar{d}_i]$, i.e. the uncertainty set S_d for the matrix D is given by:

$$S_d = \left\{ D : D = diag(d), d_i \in [d_i, \bar{d}_i], i = 1, \ldots, n \right\}.$$

It is assumed that the vector of the residual returns ε is independent of the vector of the factor returns f, and that the variance of μ is zero. The statistical properties of the estimate V lead to an uncertainty set of kind:

$$S_v = \{V : V = V_0 + W, \|W_i\|_G \leq \rho_i, i = 1, \ldots, N\}$$

where W_i denotes the i-th column of W, and $\|w\|_G = \sqrt{w^T G w}$ is the Euclidean (elliptic) norm of w with respect to a symmetric positive definite matrix G [1]. The mean return vector μ is assumed to lie in the uncertainty set S_m given by:

$$S_m = \{\mu : \mu = \mu_0 + \xi, |\xi_i| \leq \gamma_i, i = 1, \ldots, n\},$$

i.e., each component of μ is assumed to lie within a certain interval.

In this way, the return of a portfolio x is defined as: $r_x = r^T x = \mu^T x + f^T V x + \varepsilon^T x$.

The robust analog of the Markowitz mean-variance optimization problem (3) is given by:

$$\begin{aligned} \min \max_{\{V \in S_v, D \in S_d\}} & x^T Q x \\ s.t. \min_{\{\mu \in S_m\}} & E(r_x) \geq R, \\ & x \in X \end{aligned} \tag{7}$$

At the same way, the robust counterpart of maximum return problem is the following (i.e the robust counterpart of (2)):

$$\begin{aligned} \max \min_{\{\mu \in S_m\}} & E(r_x) \\ \max_{\{V \in S_v, D \in S_d\}} & x^T Q x \leq \sigma \\ & x \in X \end{aligned} \tag{8}$$

For the uncertain sets S_d, S_v and S_m above defined, the robust optimization problems (7) and (8) can be reduced to second order cone programming problems (SOCP), that are computationally tractable via standard SOCP solvers.

2.3 The Sharpe ratio problem and its robust counterparts

A problem closely related to the mean-variance problems is the *Sharpe ratio optimization problem*:

$$\max \frac{\mu^T x - r_f}{\sqrt{x^T Q x}}$$

[1] A way to define G is related to probabilistic guarantees on the likelihood that the actual realization of the uncertain coefficients will lie in the ellipsoidal unceratainty set S_v. Specifically, the definition of matrix G can be based on the data used to produce the estimates of the regression coefficients of the factor model [12].

$$x \in X$$

where r_f represents the known return on a riskless asset. The Sharpe ratio, i.e., $h(x) = \frac{\mu^T x - r_f}{\sqrt{x^T Q x}}$, is thus a performance measure that evaluates the excess return per unit of risk.

Since this maximization problem has a nonlinear and nonconcave objective function, and therefore it may be difficult to solve it directly, Goldfarb and Iyengar [16] proposed an elegant argument to formulate the problem in terms of a convex minimization problem.

All is based on the observation that $e^T x = 1$ whenever $x \in X$ (e represents an n-dimensional vector of 1's) since proportions in all securities must sum 1. Therefore the Sharpe ratio $h(x)$ can be rewritten as a homogeneous function of x as follows:

$$h(x) = \frac{\mu^T x - r_f}{\sqrt{x^T Q x}} = \frac{(\mu - r_f e)^T x}{\sqrt{x^T Q x}} =: g(x) = g\left(\frac{x}{k}\right) \qquad k > 0. \tag{9}$$

The vector $\mu - r_f e$ is the vector of the returns in excess of the risk-free lending rate. When X has the form (1), it can be proved that one can replace the normalization constraint $e^T x = 1$ with the alternative normalization constraint $(\mu - r_f e)^T x = 1$ without affecting the optimal solution. In this way, maximizing the objective function $h(x)$ is equivalent to minimizing $x^T Q x$, a strictly convex quadratic function of x (assuming Q a positive definite matrix).

In [26] Koenig and Tütüncü proved that a similar reduction can be achieved even when X is not in the form in (1), as long as $x \in X$ implies $e^T x = 1$. Under this assumption, a portfolio x^* with the maximum Sharpe ratio can be found by solving the following problem:

$$\begin{aligned} &\min \ x^T Q x \\ &s.t \ (\mu - r_f e)^T x = 1 \\ &(x, k) \in X^+ \end{aligned} \tag{10}$$

where X^+ is a cone that lives in a one higher-dimensional space than X, and which is defined as follows:

$$X^+ = \left\{x \in R^n, k \in R | k > 0, \frac{x}{k} \in X\right\} \cup \{(0,0)\}. \tag{11}$$

Moreover, the normalizing constraint can be relaxed to $(\mu - r_f e)^T x \geq 1$ by recognizing that this constraint will always be tight at an optimal solution.

Based on this observation, Koenig and Tütüncü [26] proposed the following robust counterpart of the relaxed maximum Sharpe ratio problem, where $U_\mu = \left\{\mu : \mu^L \leq \mu \leq \mu^U\right\}$ and $U_Q = \left\{Q : Q \succeq 0, Q^L \leq Q \leq Q^U\right\}$ as previously defined:

$$\begin{aligned} &\min \left\{\max_{Q \in U_Q} x^T Q x\right\} \\ &(x, k) \in X^+ \end{aligned} \tag{12}$$

$$\min_{\mu \in U_\mu} (\mu - r_f e)^T x \geq 1.$$

The problem was resolved as a second order cone programming.

2.4 Alternative robust models

Traditional robust optimization is sometimes criticized for being overly conservative. At the same time, a robust optimization approach only guards against data realizations that are allowed by the given uncertainty model, while potentially becoming very vulnerable to realizations outside of the realm of the model. However, to enlarge the uncertainty set could make the problem too conservative. Furthermore, a classical robust optimization model tend to give the same weight to all possible data realizations, which may be unrealistic in practice.

To reduce this difficulty, Bienstock proposed two alternative robust models: the *histogram* model and the *ambiguous chance-constraints* model [8]. Both models are based on two different, interleaving, problems: an *implementor problem* which picks values for the decision variables of the model, and an *adversarial problem* which finds the worst-case data corresponding to the decision variables just selected by the implementor problem. The adversarial problem, in both cases, is a mixed-integer program. On the contrary, the implementor problem is, in the first case, a quadratic convex problem, while in the second it is a quadratically constrained linear program solvable using SOCP techniques.
In constructing the models, Bienstock assumes that a time series is available from which expected returns (and variance) are computed. The uncertainty models are obtained by allowing the adversary to deviate from the given distribution in a constrained manner. So, these models are data driven; in particular, the author doesn't assume that returns are normally distributed.
In the histogram model, the adversarial returns are segmented into a fixed number of categories, or bands; the distribution is obtained by employing an approximate count of the number of assets in each band. Having constructed a set of bands, one can then roughly estimate the probability that an observation will fall in any given band.
In the ambiguous chance constrained model (that is an extension of the histogram model), the implementor chooses a portfolio of assets and the adversary chooses a probability distribution for the vector of the returns from a family of distributions of data (known to the implementor). A random vector μ of returns is then selected based on the chosen probability.
The implementor wants to choose the portfolio x^* that minimizes a risk measure.

2.5 Robust models in practice

Literature is rich of computational results that compare the behavior of the solutions obtained by robust optimization techniques and the solutions obtained by standard approaches.

Some tests with simulated and real market data indicate that robust optimization, when inaccuracy is assumed in the expected return estimates, outperforms classical mean-variance optimization in terms of total excess return a large percentage of the time [9]. Furthermore, independent tests by practitioners and academics using both simulated and market data appear to confirm that robust optimization generally results in more stable portfolio weights. However, other tests have not been as conclusive The factor that accounts for much of the difference is how the uncertainty in the parameters is modeled. Therefore, finding a suitable degree of robustness and an appropriate definition of uncertainty set can have a significant impact on the portfolio performance.

Observe that, while there are efficient algorithms to solve robust portfolio allocation problems [17], [16], these algorithms find a single point on the robust efficient frontier, that can be obtained by handling the lower limit R on the expected return and the upper limit σ on the variance as parameters of the models. In order to generate the efficient frontier, Koenig and Tütüncü [26] first determined the robust efficient portfolios with respect to the lowest and the highest expected returns, then discretized the range between these two extremes to obtain a finite number of set levels of the expected return, and finally solved problem (6) for each set level of the expected return. To obtain the robust efficient portfolios with the highest and the lowest expected returns, as well as to solve problem (6) for each intermediate value, they used the saddle-point algorithm developed by Halldórsson and R.H. Tütüncü [17].

3 Robustness and risk measures

In the classical Markowitz models, the risk is measured by means of a dispersion measure such as variance or standard deviation. More recently, starting from the observation that positive and negative deviations of the returns from their mean value play a greatly asymmetric role in the investor's perception, financial practice and related theory showed increasing interest towards quantile based measures, such as Value at Risk (VaR) [22].

VaR is a quantile-based risk measure that provides information about the amount of losses that will not be exceeded with a certain probability. Mathematically, given a probability threshold $\alpha > 0$, α-VaR is defined as the minimum level γ such that the probability that the portfolio loss exceeds γ is less than or equal to $1 - \alpha$. The *portfolio α-VaR optimization problem* can then be formulated as

$$\begin{aligned} &\min \gamma \\ &s.t.\ P(f(x,y) \geq \gamma) \leq 1 - \alpha \\ &s.t.\ x \in X \end{aligned} \tag{13}$$

where $f(x, y)$ denotes the loss function when the portfolio x is choosen from the set X of feasible portfolios and y is the realization of some random events over a fixed period of time.

The constraint $P(f(x, y) \geq \gamma) \leq 1 - \alpha$ is a chance constraint which is very

difficult to handle computationally. In practice, in order to make the problem manageable, managers, frequently, assume that the asset returns are normally distributed, in which case there is a closed form expression for the probability term in the constraint (Normal VaR). In such a special case, Goldfarb and Iyengar [16] derived the robust counterpart of the Normal VaR formulation when the future returns are modelled using a factor model, by assuming that there is some errors in the estimation of the gammas and of the covariance matrix.
As we have told above, there is a significant evidence that some asset returns are not normal. There is also some support for the belief that the variances of some asset returns are not bounded (that is, they are infinite and therefore do not exist). This, unfortunately, means that portfolio allocations obtained by using the Normal VaR tend to underestimate losses. For this motivation, when one thinks of robustness related to the risk measure VaR, it is natural to consider robustness with respect to a set of possible *probability distributions* of the uncertain returns.
An approximation to the problem of minimizing the worst-case VaR over all possible probability distributions for future returns takes the form [11]:

$$\begin{aligned} &\min_{\gamma,x} \gamma \\ s.t.\ &k\sqrt{x^T Q x} - \mu^T x \leq \gamma \\ &x \in X \end{aligned} \tag{14}$$

where $k = \sqrt{\frac{\alpha}{1-\alpha}}$ is an appropriate risk factor.

We refer to the above formulation as *Worst-Case VaR*. The Worst-Case VaR portfolio allocation is still selected on the basis of the first and second moment of the portfolio returns. The important question is how conservative is this approach. Overprotecting may result in worse overall portfolio performance than not making portfolio allocation robust.

Computational studies seem to indicate that the Worst-Case VaR does not necessarily perform better than Normal VaR and other approaches for approximating VaR at the same level of α.
Unfortunately VaR, if studied in the framework of Coherent Risk measures, lacks subadditivity (and therefore convexity) [1]. An additional difficulty with VaR is in its computation and optimization. When VaR is computed by generating scenarios, it turns out to be a nonsmooth and nonconvex function of the positions in the investment portfolio. Therefore, the optimum may be not unique.
Another criticism of VaR is that it pays no attention to the magnitude of losses beyond the VaR value. This and other undesirable features of VaR led to the development of alternative risk measures. One well-known modification of VaR is obtained by computing the Conditional Value at Risk (CVaR), defined as the mean of the tail distribution exceeding VaR [24], [25].

The α-CVaR associated with portfolio x is defined as follows:

$$CVaR_\alpha(x) = \frac{1}{1-\alpha} \int_{f(x,y) \geq VaR_\alpha(x)} f(x,y)\, p(y)\, dy$$

where, as before, $f(x,y)$ denotes the loss function when the portfolio x is choosen from a set X of feasible portfolios and y is the realization of the random events, while $p(y)$ denotes the probability of the events y.
Rockafellar and Uryasev [25] showed that minimizing CVaR can be achieved by minimizing a more tractable auxiliary function without predetermining the corresponding VaR first. Their formulation of CVaR usually results in convex programs and even linear program. Thus, their work opened the door to applying CVaR to financial optimization and risk management in practice.
An attempt of robust optimization of the CVaR was proposed in [22], where the authors implemented in a robust way the bicriteria model [2] proposed by Rockafellar and Uryasev [25], obtaining a Mixed Integer Linear Programming problem. Later, they proposed a variant of the problem to make it a Linear Programming Problem, i.e., considering the integer part of assets when they are not integer value and the difference, calculated for each kind of share, between the not-integer solution and its integer part multiplied by the current price of the share must be charged to a new variable that it is denoted as 'cash', (for each kind of share). This asset does not offer any return and must be added to the value of the portfolio, at time $(t+1)$, obtained considering only the integer part of the solutions given by the model.

A variant of the CVaR problem was formulated in [27], where the concept of Worst-CVaR is introduced. Given a probability threshold $\alpha > 0$, *Worst-CVaR* is defined as follows:

$$WCVaR_\alpha(x) = \sup_{p(\cdot)\in\mathcal{P}} CVaR_\alpha(x).$$

According to such a definition, the density function p is only known to belong to a certain set $\mathcal{P}$ of distributions. The authors proved that also Worst-CVaR is a coherent risk measure. They formulated some robust counterparts under box uncertainty and ellipsoidal uncertainty sets, showing that these problems are linear programs and second-order cone programs, respectively.

Very recently [4], robust optimization for portfolio asset allocation has been merged with stochastic approaches studying problems of the form

$$\inf_{p\in\mathcal{P}} E_p[f(x,u)]$$

where $f(x,u)$ is a payoff function which depends on decision variables x and uncertain parameters u. In these models, instead of assuming the exact knowledge of the distribution of the asset returns, some measures of robustness are

[2] In which the goal is to form a portfolio in which expected return is maximized, while some index of risk is minimized.

suggested against the distribution variation. Moreover, in the so-called *soft robust approach*, mathematical models and related approaches are proposed which are based on the preferences of the decision maker, where different guarantees are provided for different distribution probabilities. These ideas are connected closely to the theory of convex risk measures [15].
A direct connection between robust optimization and convex risk measures has been explored Natarajan et al. [19], but for the much smaller class of coherent risk measures. For future works, it would be interesting to focus more on relation between robustness and convex risk measure.

The original publication is available at *www.springerlink.com*.

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