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# A Compact Variant of the QCR Method for 0-1 Quadratically Constrained Quadratic Programs

Laura Galli Adam N. Letchford

ADDRESS: Largo B. Pontecorvo 3, 56127 Pisa, Italy. TEL: +39 050 2212700 FAX: +39 050 2212726

# A Compact Variant of the QCR Method for 0-1 Quadratically Constrained Quadratic Programs

Laura Galli* Adam N. Letchford†

**Abstract**

*Quadratic Convex Reformulation* (QCR) is a technique that was originally proposed for 0-1 quadratic programs, and then extended to various other problems. It is used to convert non-convex instances into convex ones, in such a way that the bound obtained by solving the continuous relaxation of the reformulated instance is as strong as possible.

In this paper, we focus on the case of *0-1 quadratically constrained quadratic programs*. The variant of QCR previously proposed for this case involves the addition of a quadratic number of auxiliary continuous variables. We show that, in fact, at most one additional variable is needed. Some computational results are also presented.

**Keywords**: *Combinatorial optimization, Semidefinite programming, Quadratically constrained quadratic programming*

## 1 Introduction

It has been known for some time that semidefinite programming (SDP) can be used to derive strong convex relaxations of various hard quadratic optimisation problems. This includes, for example, *zero-one quadratic programming* (0-1 QP) [11, 15], non-convex *quadratically constrained quadratic programming* (QCQP) [9, 16, 19] and 0-1 QCQP [7, 13, 17].

In a recent paper, Billionnet *et al.* [5] proposed to use SDP to *reformulate* 0-1 QP instances, rather than merely *relax* them. Their method, called *Quadratic Convex Reformulation* (QCR), has two effects. First, it converts non-convex instances into convex ones. Second, when applied to instances that are already convex, it improves the bound obtained by solving the continuous relaxation of the instance. Once QCR has been applied, the reformulated instance can be fed into any software package capable of solving convex 0-1 QP instances.

*Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56124 Pisa, Italy. E-mail: `Laura.Galli@di.unipi.it`

†Department of Management Science, Lancaster University, Lancaster LA1 4YW, United Kingdom. E-mail: `A.N.Letchford@lancaster.ac.uk`

Recently, Billionnet *et al.* [3, 4] have extended QCR to *mixed-integer quadratic programming* (MIQP) and *mixed-integer quadratically constrained quadratic programming* (MIQCQP). The purpose of the present paper is to present a much simpler variant of QCR for the case of *0-1 quadratically constrained quadratic programming* (0-1 QCQP).

The structure of the paper is as follows. In Section 2, we review the relevant literature. In Section 3, we present our extension of QCR. In Section 4, we present some preliminary computational results. Finally, some concluding comments are given in Section 5.

# 2 Literature Review

We now review the relevant literature. We cover SDP and Lagrangian relaxations for non-convex QCQP in Subsection 2.1, their application to 0-1 QP in Subsection 2.2, and the QCR method in Subsection 2.3.

## 2.1 Relaxations of non-convex QCQP

A general instance of QCQP can be written in the following form:

$$\begin{array}{ll} \inf & x^T Q^0 x + c^0 \cdot x \\ \text{s.t.} & x^T Q^j x + c^j \cdot x \leq h_j \quad (j = 1, \ldots, m) \\ & \qquad x \in \mathbb{R}^n, \end{array} \tag{1}$$

where the $Q^j$ are symmetric matrices of order $n$, the $c^j$ are $n$-vectors and the $h_j$ are scalars. (We write 'inf' rather than 'min' because it is possible that the infimum is not attainable.)

Now suppose that at least one of the $Q^j$ is not positive semidefinite (psd), so that the problem is not convex. We can derive a SDP relaxation as follows [9, 16, 19]. We define the $n \times n$ matrix $X = xx^T$, along with the augmented matrix

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Note that $Y$ is symmetric and psd. The following SDP is therefore a relaxation of non-convex QCQP:

$$\begin{array}{ll} \inf & Q^0 \bullet X + c^0 \cdot x \\ \text{s.t.} & Q^j \bullet X + c^j \cdot x \leq h_j \quad (j = 1, \ldots, m) \\ & \qquad Y \succeq 0. \end{array}$$

Here, $Q^j \bullet X$ denotes $\sum_{i=1}^n \sum_{k=1}^n Q^j_{ik} X_{ik}$, and $Y \succeq 0$ means that $Y$ is symmetric and psd.

There is a connection between SDP and Lagrangian relaxations of non-convex QCQP [8, 9, 13, 15]. Suppose we relax the constraints (1) in Lagrangian fashion, using a vector $\lambda \in \mathbb{R}^m_+$ of Lagrangian multipliers. The Lagrangian is:

$$f(x, \lambda) \;=\; x^T \left( Q^0 + \sum_{j=1}^m \lambda_j Q^j \right) x + \left( c^0 + \sum_{j=1}^m \lambda_j c^j \right) \cdot x - \sum_{j=1}^m \lambda_j h_j.$$

The relaxed problem is:

$$\inf \{ f(x, \lambda) : \ x \in \mathbb{R}^n \},$$

which is an unconstrained quadratic minimisation problem. The Lagrangian dual is:

$$\sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} f(x, \lambda).$$

As explained in [9, 13, 15], if the supremum is attainable by some multiplier vector $\lambda^*$, then $\lambda^*$ must be an optimal dual solution to the SDP. If in addition strong duality holds for the SDP, then the semidefinite and Lagrangian bounds will be equal. An analogous result holds when quadratic equations, rather than inequalities, are present.

## 2.2 Relaxations of 0-1 QP

An instance of 0-1 QP can be written in the following form:

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & Ax = b \qquad (2) \\ & Dx \leq f \qquad (3) \\ & x \in \{0, 1\}^n, \end{aligned}$$

where $Q$ is again a symmetric square matrix and $A$, $D$, $c$, $b$ and $f$ are matrices and vectors of appropriate dimension.

As observed by many authors (e.g., [10, 12, 13, 14, 15]), the condition that $x$ be binary is equivalent to the non-convex quadratic constraints

$$x_i^2 - x_i = 0 \qquad (i = 1, \ldots, n).$$

That is, 0-1 QP can be regarded as a special case of non-convex QCQP. This observation suggests immediately the following SDP relaxation:

$$\inf \left\{ Q \bullet X + c^T x : \ (2), \ (3), \ x = \operatorname{diag}(X), \ Y \succeq 0 \right\}.$$

The SDP can be strengthened using some ideas presented in [14, 18]. Given a linear equation in the system (2), say $a^j \cdot x = b_j$, and any variable, say $x_k$, the quadratic equation $(a^j \cdot x) x_k = b_j x_k$ is satisfied by all feasible solutions. This implies that the equation

$$\sum_{i=1}^{n} a_i^j X_{ik} - b_j x_k = 0 \qquad (4)$$

can be added to the SDP. In a similar way, any linear inequality in the system (3) can be multiplied by either $x_k$ or $1 - x_k$ to yield valid quadratic inequalities, which can also be converted into valid inequalities for the SDP. For further ways of strengthening the SDP, see, e.g., [8, 11, 13, 15, 17].

It follows from the result mentioned in Subsection 2.1 that, for any SDP relaxation of 0-1 QP, there is a corresponding Lagrangian relaxation. Examples of such relaxations appear, for example, in [9, 12, 13, 15].

### 2.3 The QCR method

The QCR method has its roots in an early paper by Hammer & Rubin [10]. They proposed to convert non-convex 0-1 QP instances into convex ones, simply by subtracting $\sum_{1\leq i\leq n} \alpha(x_i^2 - x_i)$ from the objective, where $\alpha$ is the minimum eigenvalue of $Q$. Note that this reformulation leaves the cost of every feasible solution unchanged.

Billionnet & Elloumi [2] applied a similar idea to unconstrained 0-1 QP. They perturbed the objective by adding terms of the form $\lambda_i(x_i^2 - x_i)$, where $\lambda \in \mathbb{R}^n$, in such a way that (i) the resulting objective function is convex, and (ii) the lower bound obtained by solving the continuous relaxation of the instance is maximised.

Billionnet *et al.* [5] extended this approach to the case of equality-constrained 0-1 QP. They perturbed the objective not only by adding terms of the form $\lambda_i(x_i^2 - x_i)$, but also by adding terms of the form $M_{jk}(a^j \cdot x - b_j)x_k$, for some constraint index $j$ and variable index $k$. To do this, they proposed to solve the SDP mentioned in the previous subsection, strengthened with the constraints of the form (4), and then set $\lambda$ and $M$ to the optimal dual values for the constraints $\mathrm{diag}(X) = x$ and (4), respectively. This is the original version of QCR.

As mentioned in the introduction, Billionnet *et al.* [3, 4] extended QCR to instances of MIQP and MIQCQP satisfying certain technical conditions. For the sake of brevity, we do not go into further details. We remark however that their approaches involve the addition of $\mathcal{O}(n^2)$ variables. Our approach to 0-1 QCQP works with the original $n$ variables only.

We remark that some alternative heuristics for selecting the multipliers $\lambda_i$ appear in [1]. They involve either minimising the trace of the perturbed matrix, or minimising the maximum eigenvalue. Also, some alternative approaches to MIQCQP are surveyed in [7].

## 3 Theoretical Results

In this section, we show how to extend QCR to general a 0-1 QCQP instance of the form:

$$\begin{aligned}
\min \quad & x^T Q^0 x + c^0 \cdot x & & (5)\\
\text{s.t.} \quad & x_i^2 - x_i = 0 & (i = 1, \ldots, n) & (6)\\
& x^T Q^j x + c^j \cdot x = h_j & (j = 1, \ldots, m). & (7)\\
& x^T Q^j x + c^j \cdot x \leq h_j & (j = m+1, \ldots, m+r). & (8)
\end{aligned}$$

For simplicity of notation, we assume that any linear constraints, or any additional valid quadratic constraints generated from them (as in Subsection 2.2), are already included in the systems (7) and (8).

It turns out that quadratic inequalities are harder to handle than quadratic equations. We therefore deal with equations only in Subsection 3.1, before moving on to the general case in Subsection 3.2.

### 3.1 The equality-constrained case

Consider an equality-constrained 0-1 QCQP instance of the form (5)-(7). The continuous relaxation of this instance is obtained by replacing the binary conditions (6) with the weaker

conditions $0 \leq x_i \leq 1$ for all $i$.

We wish to 'convexify' the instance, which in this context means transforming the instance into an equivalent 0-1 QCQP instance (which will turn out to be inequality-constrained) that has the following four properties:

- The objective function is convex.
- The constraint functions are all convex.
- The set of feasible solutions is unchanged.
- The cost of each feasible solution is unchanged.

Ideally, we would like the lower-bound from the continuous relaxation of the transformed instance to equal the lower bound from the following SDP relaxation:

$$\begin{aligned} \inf \quad & Q^0 \bullet X + c^0 \cdot x && (9) \\ \text{s.t.} \quad & \operatorname{diag}(X) = x && (10) \\ & Q^j \bullet X + c^j \cdot x = h_j \quad (j = 1, \ldots, m) && (11) \\ & Y \succeq 0. && (12) \end{aligned}$$

Moreover, in contrast to Billionnet *et al.* [4], we would like to do this without using additional variables.

Observe that, when perturbing the objective function, we can add terms of the following two forms:

- $x^T \operatorname{Diag}(\lambda) x - \lambda^T x$ for some $\lambda \in \mathbb{R}^n$
- $\mu_j (x^T Q^j x + c^j \cdot x - h_j)$ for some $\mu \in \mathbb{R}^m$.

The following theorem shows that, using these two sources of perturbation, the desired reformulation can be obtained:

**Theorem 1** *Let a 0-1 QCQP instance of the form (5)-(7) be given. Suppose that strong duality holds for the SDP (9)–(12), and that a dual optimal solution exists. Let $(\lambda^*, \mu^*)$ be such a solution, where $\lambda$ and $\mu$ are the dual variables for the constraints (10)–(11). Then, suppose we perform the following three operations:*

- *perturb the objective function of the 0-1 QCQP instance by adding terms of the form $x^T \operatorname{Diag}(\lambda^*) x - (\lambda^*)^T x$ and $\mu_j^* (x^T Q^j x + c^j \cdot x - h_j)$ for $j = 1, \ldots, m$;*
- *replace each of the quadratic equations (7) with two quadratic inequalities of opposite sign;*
- *convexify the resulting quadratic inequalities using any desired method (such as the minimum eigenvalue method).*

*Then the reformulated 0-1 QCQP instance will be convex, and the lower bound from its continuous relaxation will equal the SDP lower bound.*

**Proof.** From the equivalence of semidefinite and Lagrangian relaxations described in Subsection 2.1, the lower bound from the SDP will be identical to the lower bound from the Lagrangian dual under the stated conditions. Moreover, the optimal Lagrangian multipliers for the constraints (6)–(7) will be equal to $\lambda^*$ and $\mu^*$, respectively. So, the lower bound from the Lagrangian dual will be equal to:

$$\min \left\{ x^T \bar{Q} x + \bar{c} \cdot x + \bar{h} : \; x \in \mathbb{R}^n \right\}, \tag{13}$$

where

$$\begin{aligned} \bar{Q} &= Q^0 + \mathrm{Diag}(\lambda^*) + \sum_{j=1}^{m} \mu_j^* Q^j \\ \bar{c} &= c^0 - \lambda^* + \sum_{j=1}^{m} \mu_j^* c^j \\ \bar{h} &= -\sum_{j=1}^{m} \mu_j^* h_j. \end{aligned}$$

Now observe that the objective function of (13) is identical to the objective function of the reformulated instance. Therefore, the lower bound from the continuous relaxation is at least as large as (13). The continuous relaxation has some constraints that are not present in (13); namely, the convexified quadratic inequalities and the constraint $x \in [0,1]^n$. But these constraints cannot improve the bound, since they are convex and are implied by constraints that have already been incorporated into the objective function of (13) with optimal multipliers.

It remains to be shown that the reformulated instance is convex. Since $\lambda^*$ and $\mu^*$ belong to a feasible dual solution, we have $\bar{Q} \succeq 0$, and therefore the objective function is convex. Moreover, the constraints are convex by construction. ■

One can easily show that the dual of the SDP satisfies the Slater condition, which implies that strong duality holds. On the other hand, a dual optimal solution is not guaranteed to exist (see [8]). Fortunately, we have found that such a solution can always be found in practice.

## 3.2 The inequality-constrained case

We now move on to the general case, in which *inequalities* may be present. The SDP for this case is obtained simply by adding the following constraints to (9)–(12):

$$Q^j \bullet X + c^j \cdot x \leq h_j \qquad (j = m+1, \ldots, m+r). \tag{14}$$

It turns out, however, that we cannot always reformulate an inequality-constrained 0-1 QCQP instance in such a way that the lower bound from the continuous relaxation is equal to the SDP bound. In fact, the best possible lower bound can be *much worse* than the SDP bound. This is illustrated by the following example:

**Example:** Consider the following 0-1 QCQP instance:

$$\begin{aligned} \min \quad & -\textstyle\sum_{i=1}^n x_i \\ \text{s.t.} \quad & x_i^2 - x_i = 0 \quad (i = 1, \ldots, n) && (15) \\ & x_i x_j \leq 0 \quad (1 \leq i < j \leq n). && (16) \end{aligned}$$

The optimal profit is $-1$, and the lower bounds from the primal and dual SDPs are easily shown to be $-1$ as well. As for QCR, there is no point perturbing the objective function using the equations (15), since negative values for the multipliers $\lambda_i$ would destroy convexity and positive values would weaken the lower bound. On the other hand, the quadratic inequalities (16) are non-convex, and therefore must be convexified using the equations (15). By symmetry, there exists an optimal reformulation in which the inequalities (16) are replaced by inequalities of the form:

$$\alpha(x_i^2 - x_i) + \alpha(x_j^2 - x_j) + x_i x_j \leq 0 \qquad (1 \leq i < j \leq n),$$

for some real $\alpha$. For convexity, we require $\alpha \geq 1/2$. The best lower bound is obtained when $\alpha = 1/2$. Then, the optimal solution $x^*$ to the continuous relaxation is $(1/2, \ldots, 1/2)^T$, yielding a lower bound of $-n/2$. □

This example also illustrates the fact that, to find an optimal reformulation, one needs to find an optimal perturbation of each quadratic inequality, and the objective function, simultaneously. It can be shown that is possible to find these perturbations by solving a large SDP, involving a matrix variable of order $(n+1)r$. We do not go into details, however, since we do not recommend such an approach.

A natural alternative way to handle quadratic inequalities is to convert them into equations, by adding (continuous and non-negative) slack variables. One can then apply the reformulation scheme presented in the previous subsection. It can be shown that, if this is done, the lower bound for the resulting *mixed* 0-1 QCQP instance is equal to the SDP bound for the original 0-1 QCQP instance. This does however mean that $r$ new variables have to be added.

In fact, it is possible to use just *one* slack variable, as shown in the following proposition:

**Proposition 1** *Let a 0-1 QCQP instance of the form (5)-(8) be given, and suppose as before that a dual optimal SDP solution $(\lambda^*, \mu^*, \nu^*)$ exists. Then, suppose we perform the following four operations:*

- *add the following constraint to the 0-1 QCQP instance:*

$$\sum_{j=m+1}^{m+r} \nu_j^* (x^T Q^j x + c^j \cdot x) + s = \sum_{j=m+1}^{m+r} \nu_j^* h_j, \tag{17}$$

  *where $s$ is a new continuous and non-negative slack variable;*

- *perturb the objective function of the resulting mixed 0-1 QCQP instance by adding the variable $s$, along with terms of the form $x^T \mathrm{Diag}(\lambda^*) x - (\lambda^*)^T x$, $\mu_j^*(x^T Q^j x + c^j \cdot x - h_j)$ for $j = 1, \ldots, m$, and $\nu_j^*(x^T Q^j x + c^j \cdot x - h_j)$ for $j = m+1, \ldots, m+r$;*

- *replace each quadratic equation with two quadratic inequalities;*
- *convexify all quadratic inequalities using any desired method.*

*Then the reformulated mixed 0-1 QCQP instance will be convex, and the lower bound from its continuous relaxation will be equal to the SDP bound for the original 0-1 QCQP instance.*

**Proof.** Observe that the equation (17) is obtained by multiplying each of the $r$ quadratic inequalities (8) by its corresponding $\nu^*$ value, summing the resulting inequalities together, and adding a slack variable. Since $\nu^*$ is by assumption part of a dual optimal solution, the SDP bound for the original instance would remain unchanged if we replaced the $r$ inequalities (8) with the single equation (17). The rest of the proof is similar to that of Theorem 1. ■

# 4 Preliminary Computational Results

We now present the results of some preliminary computational experiments. We created six random 0-1 QCQP instances of the following form:

$$\max\left\{c^0 \cdot x + x^T Q^0 x : \; c^1 \cdot x + x^T Q^1 x \leq b, \; x \in \{0,1\}^n\right\},$$

where $c^0$, $c^1$, $Q^0$ and $Q^1$ all have positive integer components, and $b$ is set to:

$$\left\lceil \frac{1}{2}\sum_{i=1}^{n} c_i^1 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}^1 \right\rceil.$$

The instances have $n \in \{5, 10, \ldots, 30\}$. Note that, by construction, the optimal solutions to such instances have a positive profit.

For these instances, we applied the following five reformulation schemes:

1. Convexify the objective and constraint independently, using the *minimum eigenvalue* method in [10].
2. As above, but using the *minimum trace* method in [1].
3. As above, but using the *min-max eigenvalue* method in [1].
4. Best reformulation *without* the slack variable (see Subsection 3.2).
5. Best reformulation *with* slack variable (see Subsection 3.2).

The eigenvalues were computed using the C function `Jacobi_Cyclic_Method`, which is available in the Mathematics Source Library at:

`http://www.mymathlib.com/matrices/eigen/symmetric.html`

The SDPs were solved using the routine `CSDP`, due to Borchers [6], which is available as part of COIN-OR at:

`http://projects.coin-or.org/Csdp`

We also solved the instances to proven optimality using IBM CPLEX v. 12.5. (On our machine, we were unable to solve larger instances to optimality.)

Table 1 shows, for each instance and each scheme, the gap between the upper bound and the optimum, expressed as a percentage of the optimum. We do not show times because, for all schemes, the time taken to compute the upper bound (fractions of a second) was negligible compared to the time taken to solve the 0-1 QCQP instance to optimality (minutes or hours).

Table 1: Percentage integrality gaps of five reformulation schemes

| $n$ | % gap1 | % gap2 | % gap3 | % gap4 | % gap5 |
|---|---|---|---|---|---|
| 5 | 133.96 | 116.98 | 119.36 | 116.98 | 44.37 |
| 10 | 82.43 | 74.90 | 76.13 | 74.90 | 18.34 |
| 15 | 59.20 | 52.99 | 54.59 | 52.99 | 5.53 |
| 20 | 69.29 | 64.65 | 66.19 | 64.65 | 5.03 |
| 25 | 64.65 | 62.34 | 63.53 | 62.34 | 6.77 |
| 30 | 66.03 | 63.02 | 63.65 | 63.01 | 3.57 |

It is apparent that the 'minimum trace' bound is slightly better than the 'min-max-eigenvalue' bound, which in turn is slightly better than the 'minimum eigenvalue' bound. Also, the best bound obtainable without a slack variable is only very slightly better, on one instance, than the 'minimum trace' bound. The bound obtained with the slack variable, on the other hand, is much stronger. This is in accordance with the results in Subsection 3.2.

We wanted also to compare the effect of these five options within a branch-and-bound context. Unfortunately, CPLEX seems to perform an internal reformulation scheme of its own prior to applying branch-and-bound. The user is not given the option to switch this feature off, and therefore we were unable to make the desired comparison.

# 5 Conclusion

In this paper, we have presented a new and simpler variant of QCR for zero-one quadratically constrained quadratic programs. For the equality-constrained case, our method works with the original variables, rather than adding $\mathcal{O}(n^2)$ variables as in [4]. For the general case, we have shown that it is still possible to work with the original variables, but one can obtain a significantly stronger relaxation by adding a single continuous slack variable.

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