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A multicriteria method for a plurality of deciders

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Abstract

In this Technical Report we present a method that can be used by a set of decision makers (or deciders) as a decision aiding tool for the ranking of a certain number of alternatives according to a given set of criteria. The method aims at producing a directed multigraph involving all the alternatives (as nodes of the multigraph) so that it is possible for the deciders to identify the worst alternatives and the best alternatives. The worst alternatives are never selected by the deciders that perform their final selection among the best alternatives.
Comments, observations and reports of errors are very welcome.

Keywords: *multicriteria, multideciders, decision-aiding tools*

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1 Introduction

In this Technical Report we present a method that can be used by a set of decision makers (also termed *deciders*) for the ranking of the **a** alternatives of the set A according to the criteria of the set C composed of **c** elements.
According to a classical approach of the multicriteria methods ([2], [11], [6], [8]) the proposed method aims at providing the deciders an aiding tool for the taking of their decisions rather than a tool that provides them with a single and well defined best alternative although in some cases this may be the effective outcome of the method itself.
The literature abounds in multicriteria methods though in many cases ([8]) they are designed for a single decider or are reduced to voting methods ([14], [15]) through the setting up of a correspondence between criteria and voters and between alternatives and candidates. In all these cases it is assumed that a final total ranking with possible ties among the alternatives is produced and such ranking assumed to satisfy the property of transitivity. In other cases (as it may occur with Electre, [2], [6]) we have methods that do not aim at producing a total order of the alternatives but rather at producing a subset of equivalent and incomparable best alternatives.
Our method belongs to this second family and has been designed as a multicriteria multideciders method through the use of a binary relation that is not transitive by definition and that aims at producing individual partial orders among the alternatives to be merged in a collective multigraph. The multigraph is the outcome of the process and represents a visual aid to the deciders for their reaching a final decision about the alternatives of the set A. More on all this shortly.
Before going on with a description of the basic elements of our method (see section 2) we wish to say something about its manipulability or about its fragility to strategic behaviors. In general voting methods ([14], [15]) as multicriteria multideciders methods[1] are manipulable otherwise they would coincide with a dictatorship[2]. This result derives from the the Gibbard-Satterthwaite theorem and assures us ([14], [15]) that with three or more candidates every non dictatorship aggregation method is manipulable or is not strategy proof. With this we mean that through strategic voting (or by misrepresenting their preferences) some voters may obtain a better outcome than by declaring truthfully their preferences.
Our method is not an aggregation method in the Arrow's meaning (since it is based on a non transitive binary relation and produces possibly cyclical multigraphs). Anyway it may suffer the same problem although we think that the deciders do not really have any gain from behaving strategically since they usu-

[1] A voting method can be seen as a multicriteria multideciders method since every voter/decider ranks the candidates according to a set of possibly individual criteria in order to produce a ranking that is usually assumed to be both complete and transitive. Such rankings are, at a successive stage, merged in a single social ranking that is assumed, in its turn, to be both total and transitive.

[2] Of course we are assuming that the hypotheses of the Gibbard-Satterthwaite theorem are satisfied ([14], [15]).

ally do not know the preferences of the other deciders and, moreover, each of them is involved in a multi-steps process that, in a certain sense, masks the effects of a strategic voting and prevents each decider from being sure to have a gain from behaving in that way. With this said we note how the verification of what we have asserted is one of the open problems that we think are worth of a deeper investigation (see section 9).
The proposed method has been nicknamed *merawti* ([3], [5]) as a *me*thod for the *r*anking of *a*lternatives *w*ith *ti*es and this name will be used as a reference throughout the rest of this Technical Report.
As we show in section 6 *merawti* is not perfect ([14], [15], [16]) and there are situations in which it may fail although it keeps its validity as a tool for the gaining, from the deciders, of a deeper knowledge of the relations among the available alternatives according to the criteria of the set C.

2 The basic elements

2.1 Deciders, criteria and alternatives

We assume that the *deciders* involved in the process act as peers with neither subordination relations nor veto powers nor reciprocal constraints ([9], [10]). According to a classical approach (see for instance [14] and [15]) and in order to avoid any garbage-in garbage-out effect we require that the deciders are not "irrational" and therefore that they have acyclic preferences among the alternatives according to the whole set C of the criteria. With this we mean that every decider d_i defines an **individual order** on the elements of the set A. Relaxing a little bit what is usually assumed in the literature we admit that an individual order, based on a particular order to be defined shortly[3], must be acyclic but may be partial so to contain pair of alternatives that are incomparable among themselves, at least according to the criteria of the set C. We define this type of decider as **weakly rational**. If a decider has an acyclical total ordering of the alternatives we define him as simply **rational**.
We devote section 4 to show how an "irrational" decider (or a decider with cyclic preferences) can use the particular order in order to revise his evaluations so to provide acyclic preferences described by an acyclic graph.
The *criteria* of the set C are assumed to have the same weight or importance. If this was not true we could impose a lexicographic ordering on the criteria and use this ordering for the filtering of the alternatives in a succession of steps. We do not deny the importance of the lexicographic approaches rather we believe that they cannot be applied in our context. Under our hypothesis each criterion is assumed to produce a total ordering of the alternatives with possible ties. The aim of the proposed method is the merging of such orderings so to produce

[3] An individual order is a particular order on which we impose the requirement that the associated directed graph is acyclic and made only of one connected component. We recall that a directed graph is acyclic if it does not contain any closed directed path or a path that starts and ends at the same node ([13]) otherwise it is termed cyclic. Further details will be provided shortly.

a directed multigraph MG or a graph where, among two nodes, we can have more than one directed arc.
Last but not least, the *alternatives* represent the object of the choice and form an homogeneous set A of ex-ante equivalent elements. At the end of the whole procedure the alternatives end up as grouped in three disjoint sets: the set $\hat{A}$ of the best alternatives, the set $\check{A}$ of the worst alternatives and the set of the neutral alternatives. Those of the first set are incomparable among themselves and are usually preferred to all the others, those of the second set are incomparable among themselves and are not preferred to any other alternative whereas those of the last set have both more preferred and less preferred alternatives.
In the basic version of the proposed method both sets A and C are assumed to be exogenously assigned to the deciders so to be part of their common knowledge. We devote section 8 to show how the proposed method can be used if one or both sets are defined endogenously by the deciders so that each decider[4] $d_i \in D$ has his own set of alternatives A_i or his own set of criteria C_i or both to be used in the definition of the multigraph MG.

2.2 The basic binary relations

The *merawti* method uses two families of binary relations. In the former family we have two classical binary relations with classical properties ([12], [2], [13]): an indifference relation $\sim_i$ and a strict preference relation $\succ_i$ for each criterion $c_i \in C$ and on every pair of alternatives $(a_j, a_k) \in A$. Such relations allow the definition of one total ordering (with possible ties) of the alternatives for each criterion. Every total ordering corresponds to a linear graph.
In the latter family ([3], [4]) we have only the binary relation $>$ that is defined as follows. For any pair $(a_j, a_k) \in A$ we[5] count the number of criteria $c_i \in C$ such that $a_j \succ_i a_k$ and the number of criteria $c_i \in C$ such that $a_k \succ_i a_j$. If the former is strictly greater that the latter we write $a_j > a_k$ whereas if the latter is strictly greater than the former we write $a_k > a_j$ but if the two numbers are equal we are unable to define an indifference condition between the two alternatives and rather we define the two alternatives as **incomparable**. We note how this incomparability condition prevents any reduction of the proposed binary relation to a majoritarian relation where such an equality would be interpreted as a tie between the involved alternatives ([7], [1]).
From the definition of the binary relation $>$ we derive immediately how it defines a partial order on the set A owing to the possible presence of incomparable alternatives. It is easily verified, moreover, how such relation:

[4]It is of course possible that some of the deciders have the same sets of alternatives and criteria.

[5]In many cases we use the term “we” and its derivations as referring to the author but in some cases we use it as a shorthand for “the deciders”. The context should make clear the right meaning in the various cases.

- fails transitivity[6] (since from $a_j > a_k$ and $a_k > a_l$ we cannot derive for sure, from our definition, $a_j > a_l$);

- satisfies asymmetry (since from $a_j > a_k$ we immediately derive $\neg(a_k > a_j)$).

For these reasons we nicknamed it a **particular order** ([3], [4], [5]). We note how the particular order fails transitivity from its being defined through pairwise comparisons among alternatives and by using independent criteria ([14], [15]). It is easy to see, at this point, how each decider d_i by using the foregoing relations can define a directed graph[7] $G_i = (N_i, W_i)$ that, as added requirements, must be acyclic[8] and cannot contain subgraphs or isolated nodes (see also footnote 3 and section 5) but can contain incomparable alternatives (and so it is not complete). In each graph G_i there is a directed arc $h \to j$ between two nodes $h, j \in N_i$ if the corresponding alternatives $a_h, a_j \in A$ are such that $a_h > a_j$.

2.3 Something more about transitivity

In section 2.2 we have seen how the particular order $>$ has been defined so that transitivity is not guaranteed to be verified as a general property.
The lack of any guarantee of transitivity (see also section 3 and [4]) derives from the role played by the criteria and from their being assumed as independent so that there is no guarantee that the triples of alternatives are pairwise ranked by the criteria in a way that allows transitivity to be satisfied. Anyway it may happen that a given graph G_i satisfies this property.
From the definition of the particular order $>$ ([4]) we derive, indeed, that a graph G_i satisfies transitivity if for each triple $i, j, k \in N_i$ we have that:

- each pair of nodes is connected by a directed arc;

- the three arcs are properly oriented so that if we have $i \to j$ and $i \to k$ we must have $k \to j$.

We can easily see that, in particular cases, a graph G_i can satisfy transitivity. On the other hand, for a graph G_i we speak of **local transitivity** if the foregoing conditions are satisfied only by some of the possible triples of nodes whereas, if such conditions are violated by all the triples of nodes, we speak of non transitivity.
An example of a locally transitive graph is given in Figure 2 whereas an example of a non transitive graph is given in Figure 3.
We note how three nodes pairwise connected by three directed arcs either satisfy local transitivity or are connected in a cycle so to violate transitivity. If among the three nodes we have less than three arcs we have incomparable alternatives.

[6] We recall that given a set A a binary relation R on this set is said to be transitive if and only if for any $i, j, k \in A$ from iRj and jRk we may derive iRk.

[7] In the basic version we have $N_i = A$ whereas W_i defines the set of the arcs associated to the particular order for the decider d_i.

[8] As we show in section 4 the definition of the $>$ binary relation does not, by itself, rule out the possibility that a given graph G_i is cyclic so the property of being acyclic is an added requirement for each decider.

2.4 The two main steps of the *merawti* method

The method is based on two main consecutive steps, one individual and the other collective.
The **individual step** is carried out by the single deciders in isolation. In this step each decider defines **c** total orderings of the **a** alternatives and then merges such orderings (represented as linear graphs) in a single acyclic directed graph G_i by using the particular order $>$. From the properties of $>$ we derive that the graphs G_i may be not complete so that there may be pairs of alternatives that are incomparable and so that not connected by any directed arc. On each G_i we impose that it must be acyclic and made of only one connected component.
The **collective step** is implemented with a mechanical procedure by which the graphs G_i are merged so to form a single multigraph MG where between any two nodes we can have more than one directed arc, as it is indicated by the multiplicity index on each arc.
The **merging** is easily accomplished. We firstly draw the nodes, one for each alternative. Then, for each pair of alternatives, we count one directed arc for each directed arc that appears between the same alternatives in each graph G_i keeping the orientation unchanged. Lastly we draw an arc with the same orientation between the two nodes and associate to it an integer indicating the multiplicity of that connection. In this way we have defined the multigraph MG.
If, during the process of construction, between two alternatives we have two arcs with opposite orientation we neither wipe them out nor collapse them in an undirected arc since we do not want any form of compensation between criteria that rank the two alternatives in opposite ways and since in the MG we want to represent only strict preference relations between pairs of alternatives. Once the MG has been obtained the deciders can use it to identify the set $\hat{A}$ of the best alternatives and possibly to perform the final selection of an alternative from this set. Before presenting more formally the *merawti* method and before describing its main features together with some weaknesses and possible problems we present an example ([3], [5]).

3 An example

3.1 The context

In order to give some concreteness to our argumentations we present here an example where the deciders are assumed to be weakly rational (see section 2.1). In section 4 we discuss in short how the method can be used by the single deciders in order to handle the presence of cycles in the graphs G_i.
In the current example we assume to have:

- three deciders so that $D = \{d_1, d_2, d_3\}$
- four alternatives so that $A = \{a_1, a_2, a_3, a_4\}$

- four criteria so that $C = \{c_1, c_2, c_3, c_4\}$

In the individual step each decider produces four linear graphs, each with four nodes and three links, and merges them in an acyclic directed graph G_i.
In the collective step the three graphs G_i are mechanically merged in the directed multigraph MG.

3.2 The first decider

We assume that the first decider has the following rankings of the four alternatives of the set A according to the four criteria of the set C:

$a_1 \sim_1 a_2 \succ_1 a_3 \succ_1 a_4$

$a_2 \sim_2 a_3 \succ_2 a_4 \succ_2 a_1$

$a_3 \sim_3 a_4 \succ_3 a_1 \succ_3 a_2$

$a_1 \succ_4 a_3 \succ_4 a_2 \succ_4 a_4$

Such total orderings with ties are represented in Figure 1 (from left to right and from top to bottom) where the alternatives are represented as nodes labeled with the index of each alternative whereas a strict preference is represented as a directed arc and an indifference as an undirected arc. By using the foregoing

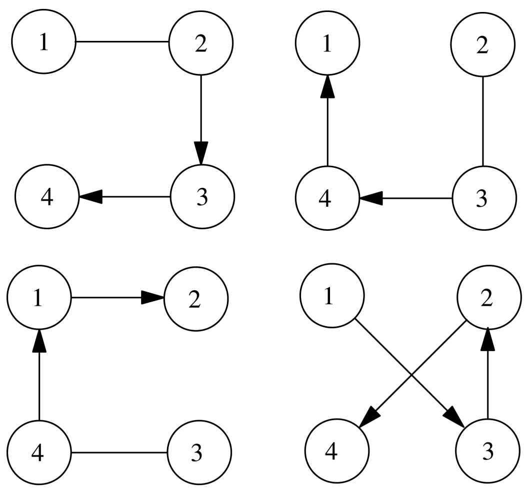

Figure 1: *The four total orderings for d_1*

rankings and performing the six pairwise comparisons among the four alternatives d_1 obtains the following results:

(a_1, a_2) $a_1 \sim_1 a_2$, $a_2 \succ_2 a_1$, $a_1 \succ_3 a_2$, $a_1 \succ_4 a_2$ or $a_1 > a_2$ with a directed arc from a_1 to a_2;

(a_1, a_3) $a_1 \succ_1 a_3$, $a_3 \succ_2 a_1$, $a_3 \succ_3 a_1$, $a_1 \succ_4 a_3$ and so no directed arc between a_1 and a_3;

(a_1, a_4) $a_1 \succ_1 a_4$, $a_4 \succ_2 a_1$, $a_4 \succ_3 a_1$, $a_1 \succ_4 a_4$ and so no directed arc between a_1 and a_4;

(a_2, a_3) $a_2 \succ_1 a_3$, $a_2 \sim_2 a_3$, $a_3 \succ_3 a_2$, $a_3 \succ_4 a_2$ or $a_3 > a_2$ with a directed arc from a_3 to a_2;

(a_2, a_4) $a_2 \succ_1 a_4$, $a_2 \succ_2 a_4$, $a_4 \succ_3 a_2$, $a_2 \succ_4 a_4$ or $a_2 > a_4$ with a directed arc from a_2 to a_4;

(a_3, a_4) $a_3 \succ_1 a_4$, $a_3 \succ_2 a_4$, $a_3 \sim_3 a_4$, $a_3 \succ_4 a_4$ or $a_3 > a_4$ with a directed arc from a_3 to a_4.

Such results are obtained under the hypothesis that both $\sim_i$ and $\succ_i$ satisfy transitivity. According to such calculations d_1 produces the directed acyclic graph G_1 of Figure 2.

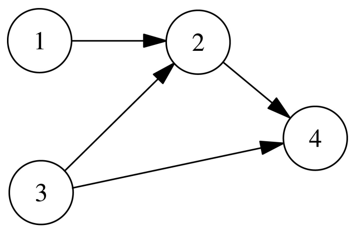

Figure 2: G_1

From this graph we see how, according to d_1, a_4 is the worst alternative (since it is represented as a node with no outgoing arcs but with incoming arcs) so that $\check{A} = \{a_4\}$ whereas the pairs (a_1, a_3) and (a_1, a_4) are incomparable (since there is no arc between such alternatives) and a_1 and a_3 are the best alternatives (since they are both represented as a node with no incoming arcs but with outgoing arcs) so that $\hat{A} = \{a_1, a_3\}$. We easily see how this graph satisfies local transitivity.

3.3 The second decider

The second decider may have the following rankings of the same alternatives of the set A according to the same criteria of the set C:

$a_1 \succ_1 a_2 \succ_1 a_3 \sim_1 a_4$

$a_2 \sim_2 a_3 \succ_2 a_4 \succ_2 a_1$

$a_3 \succ_3 a_4 \sim_3 a_1 \succ_3 a_2$

$a_1 \sim_4 a_3 \succ_4 a_2 \sim_4 a_4$

From these total orderings with ties d_2 can obtain four linear graphs similar to those of Figure 1. By performing, in his turn, the six pairwise comparisons among the four alternatives he can conclude that:

- he cannot state any preference relation, according to the binary relation $>$, between a_1 and a_2, a_1 and a_4, a_2 and a_3 so that he cannot draw any arc between the corresponding nodes;
- he can state that $a_3 > a_1$, $a_2 > a_4$, and $a_3 > a_4$ so that he can draw an arc between each pair of corresponding nodes.

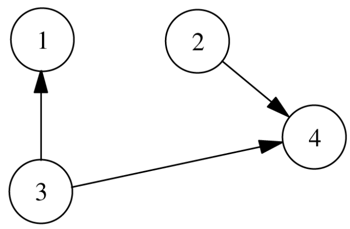

Figure 3: G_2

From such considerations he can draw his acyclic graph G_2 of Figure 3.
From his graph d_2 derives that, according to him, a_2 and a_3 are the best and incomparable alternatives (so that $\hat{A} = \{a_2, a_3\}$) whereas a_1 and a_4 are the worst and incomparable alternatives (so that $\check{A} = \{a_1, a_4\}$).

3.4 The third decider

The third decider may have the following rankings of the alternatives of the set A according to the criteria of the set C:

$a_1 \succ_1 a_2 \succ_1 a_3 \sim_1 a_4$

$a_1 \succ_2 a_3 \succ_2 a_4 \succ_2 a_2$

$a_3 \succ_3 a_2 \sim_3 a_1 \succ_3 a_4$

$a_3 \succ_4 a_1 \succ_4 a_2 \succ_4 a_4$

From these total orderings with ties d_3 can obtain four linear graphs similar to those of Figure 1. By performing, in his turn, the six pairwise comparisons among the four alternatives he can conclude that:

- he cannot state any preference relation, according to the binary relation $>$, between a_1 and a_3;

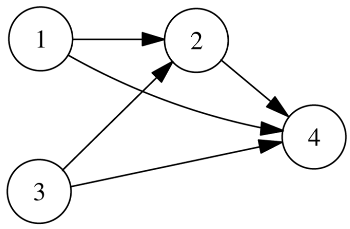

Figure 4: G_3

- he can state that $a_1 > a_2$, $a_1 > a_4$, $a_2 > a_4$, $a_3 > a_2$ and $a_3 > a_4$.

From such considerations he can draw his acyclic graph G_3 of Figure 4.
From his graph d_3 derives that, according to him, a_4 is the worst alternative (so that $\check{A} = \{a_4\}$) whereas a_1 and a_3 are the best incomparable alternatives (so that $\hat{A} = \{a_1, a_3\}$) .
We note that G_3 is locally transitive but is not transitive owing to the presence of the two incomparable alternatives.

3.5 The construction and the interpretation of the directed multigraph

When the three deciders are done and have defined their graphs G_i they can switch to the collective step and merge such graphs in the multigraph MG. In the present case the deciders obtain the multigraph of Figure 5 that is made of only one connected component and does not contain isolated nodes.

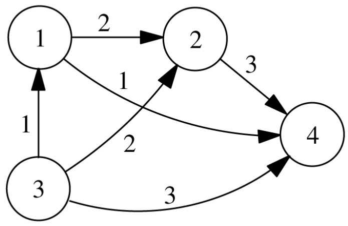

Figure 5: *The resulting MG*

Once they have defined the multigraph MG the deciders have to interpret it. In this easy case if they use[9] *forward pruning* (a sort of forward induction, see [10], [9], [11]) the deciders can remove iteratively the best alternatives (and so the nodes without incoming arcs but with outgoing arcs) so to identify a_4 as the worst alternative in this case as corresponding to the last node left[10].

[9] Both forward and backward pruning will be defined more formally in section 5.
[10] The forward succession of nodes removal in this case is $3, 1, 2$.

On the other hand, by using *backward pruning* (a sort of backward induction, see [10], [9], [11]) the deciders can remove iteratively the worst alternatives (and so the nodes without outgoing arcs but with incoming arcs) so to identify a_3 as the best alternative (as corresponding to the last node left) in this case[11].
We underline how, in this case, a_3 is strictly preferred to a_1 by only one decider (whereas the other two find these alternatives as incomparable), a_4 is unanimously evaluated worse than a_2 and a_3 and a majority of two over three deciders evaluates a_1 and a_3 better than a_2.
We note how both *forward* and *backward pruning* could be used even by the single deciders but only in cases where they act in isolation since, if they are engaged in a collective effort, it is mandatory that the graphs G_i are used as ingredients of the multigraph MG as they are and so without any pruning.
Further details will be provided in section 5.

4 How to handle "irrationality"

In this section we discuss the use of the particular order on which *merawti* is based as an aiding tool for an "irrational" decider[12] and so a decider that produces a cyclic[13] graph G_i.
For this aim we imagine a case of four alternatives and three criteria and to have one decider (that we can denote as d_4) whose preferences are as follows[14,15]:

$$a_1 \succ_1 a_2 \succ_1 a_3 \succ_1 a_4$$

$$a_2 \succ_2 a_3 \succ_2 a_4 \succ_2 a_1$$

$$a_3 \succ_3 a_4 \succ_3 a_1 \succ_3 a_2$$

It is easy to see how such profiles give rise to a cyclic graph ([14], [15]), let us denote it as G_4 (see Figure 6, (a)). From the definition of the particular order, since the number of the criteria is odd and the profiles contain only strict preference relations, we get that all the alternatives are comparable and that we have a loss of transitivity since, as we show in Figure 6, we have:

(b) $a_1 > a_2 > a_3 > a_4 > a_1$

(c) $a_1 > a_2 > a_3 > a_1$

[11] The backward succession of nodes removal in this case is $4, 2, 1$.

[12] We would like to thank one of the anonymous reviewers of a preceding version a paper from which this Techincal Report originated for having pointed out this crucial feature of the proposed method. What we discuss here is not a general solution, whose development requires further investigation and research, but represents, anyway, a good starting point.

[13] We recall that a directed graph is cyclic if it contains at least a closed directed path or a path that starts and ends at the same node ([13]).

[14] We note that we assume that the property of unrestricted domain is satisfied so we cannot rule out any preference profile of the decider ([14], [15]).

[15] In order to see how this decider can belong to the same group of deciders that we have used for the example of section 3 we refer to section 8.

(d) $a_1 > a_2 > a_4 > a_1$

but

(e) $a_2 > a_3 > a_4$ and $a_2 > a_4$

(f) $a_3 > a_4 > a_1$ and $a_3 > a_1$

so to have a local transitivity.

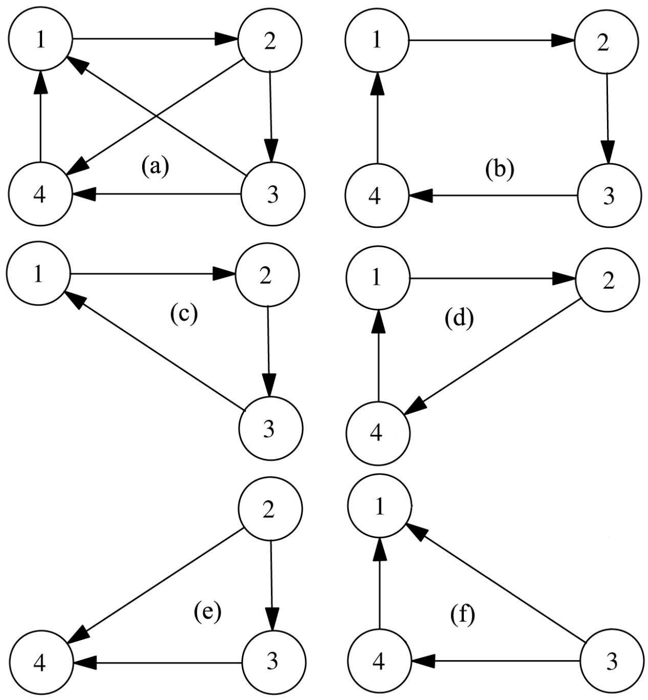

Figure 6: *The preferences of an "irrational" decider*

At this point d_4, in order to be able to merge his graph with those of the other deciders, must produce an acyclic graph. For this aim he can revise the process that led him to the definition of G_4 or he can directly act on the graph G_4 so to make it acyclic. We note how in both cases he has not to justify the adopted procedure to the other deciders that are satisfied if d_4 is able to produce an acyclic graph that best satisfies his assessment of the alternatives according to the possibly personal set of the criteria.
In the former case d_4 can iteratively revise his profiles of preferences until he is able to produce an acyclic graph.
In the latter case d_4 can try to identify the minimal set of arcs whose removal makes G_4 an acyclic graph.
If we consider the graph and the subgraphs of Figure 6 we easily see how this

minimal set is composed by the arc $1 \to 2$. The removal of this arc can be motivated as follows:

- a_2 is preferred to a_3 and a_4;
- a_3 is preferred to a_1 and a_4;
- a_4 is preferred to a_1;
- a_1 is preferred only to a_2.

In this way the local transitivities (see Figure 6 (e) and (f)) help d_4 in relaxing his preferences on a_1 so to be able to remove the arc $1 \to 2$ that makes cyclic his graph G_4. This removal is equivalent to the turning of a minimal number of strict preference relations $a_1 \succ_i a_2$ into indifference relations $a_1 \sim_i a_2$ so to make incomparable these two alternatives, according to the definition of the particular order $>$.

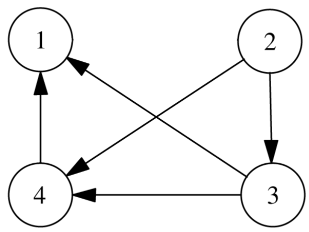

Figure 7: *G_4 made acyclic*

Once the arc $1 \to 2$ has been removed from the cyclic graph G_4 we get the new acyclic version of figure 7 where we have that, according to d_4:

$\hat{A} = \{a_2\}$

$\check{A} = \{a_1\}$

since the alternatives a_1 and a_2 have been made incomparable among themselves.
This removal procedure was easily applied in this case but in more complex cases it may prove inapplicable so that the only solution for d_4 is the adoption of the revision procedure that we briefly hinted in one of the foregoing paragraphs.

5 A description of the method

We have now all the necessary ingredients to describe more formally the *merawti* method as a decision aiding tool for a certain number **d** of deciders ([3], [5]). The method is based on the following steps:

- an individual step,

- a collective step,
- a final step.

During the **individual step** each decider d_i produces **c** total orderings with possible ties of the alternatives and then merges them in a directed graph G_i using the particular order $>$. As we have seen in section 2.1, every graph G_i defines an individual order and is characterized by the following defining properties:

- is acyclic as an explicit requirement,
- is partial from the definition of $>$,
- is made of a single connected component as a further explicit requirement.

With the last requirement we essentially aim at avoiding the presence in a graph G_i of isolated nodes that would correspond to alternatives that are incomparable with both the best and the worst alternatives of a G_i (otherwise a node would not be isolated).
In the **collective step** the **d** graphs G_i are merged in a single multigraph MG so that the alternatives of the set A are partitioned in the disjoint subsets of the best, worst and neutral alternatives.
Once the multigraph MG has been produced its topological structure depends on the structure of its composing graphs G_i. From the properties that we have imposed on the G_is we derive that MG cannot contain isolated nodes and indeed it is composed by a single connected component.
At this point we can try to identify the sets $\hat{A}$ and $\check{A}$ and use both *forward pruning* and *backward pruning* as analysis tools to verify the composition of such sets.
If $MG = \{N, W\}$ we can define *forward pruning* as follows:

(1) with N_O we identify the nodes without incoming arcs but with outgoing arcs and with W_O the arcs among the nodes of N_O and those of $N \setminus N_O$;

(2) we evaluate[16]:

$$N = N \setminus N_O$$
$$W = W \setminus W_O$$

(3) if from N we can identify a new set N_O we repeat the procedure from step (2) otherwise the procedure is over.

The requirement on N_O prevents the procedure from eliminating isolated nodes that have no incoming arcs and also no outgoing arcs. When the procedure stops we can have:

[16]We note how a relation such as $a = a \oplus b$ for any binary operation $\oplus$ must be read as an assignment relation by which a takes as its new value its old value modified with the value of the variable b through the binary operation $\oplus$.

- isolated nodes that surely belongs to $\check{A}$,
- nodes connected in cycles that do not belong to $\check{A}$ although, as we show in section 6, we may have an unbalancing of the multiplicity of the involved arcs that allow us to identify some relatively worst alternatives.

On the other hand *backward pruning* can be defined as follows:

(1) with N_I we identify the nodes without outgoing arcs but with incoming arcs and with W_I the arcs among the nodes of N_I and those of $N \setminus N_I$;

(2) we evaluate:

$$N = N \setminus N_I$$
$$W = W \setminus W_I$$

(3) if from N we can identify a new set N_I we repeat the procedure from step (2) otherwise the procedure is over.

The requirement on N_I prevents the procedure from eliminating isolated nodes that have no outgoing arcs and also no incoming arcs. When the procedure stops we can have:

- isolated nodes that surely belongs to $\hat{A}$,
- nodes connected in cycles that do not belong to $\hat{A}$ although we may have an unbalancing of the multiplicity of the involved arcs that allow us to identify some relatively best alternatives.

The key point, once the multigraph MG has been constructed, is however represented by the cardinality of the set $\hat{A}$ rather than by the pruning methods that are nothing more than ancillary fallible analysis methods. If this set is empty (and so it has a null cardinality) the deciders must face this situation as it is described in section 6. If, on the other hand, this set is not empty (and so it has a positive cardinality) the deciders can switch to the final selection step that is described in section 7.

6 Possible problems and failures

As every decision tool our method is not perfect and so there are situations that may cause troubles and may let us question both its applicability and its validity ([5],[14], [15], [6]).
The main problems with the method occur whenever the deciders are not able to define a non empty set $\hat{A}$. The condition $\hat{A} = \emptyset$ can occur if the MG has no node without incoming arcs. A situation of this type is represented in the left side of Figure 8 in the case of four alternatives and at least three deciders. In situations like this the deciders can profitably use a Condorcet like approach ([14], [15]) and so can pairwise compare the alternatives so to define, for each pair, the winning and the losing alternative. This approach works as follows:

- the deciders consider the labels on the arcs of the MG as the votes for one alternative against the other;

- they declare a winning alternative in each pair by selecting the alternatives that gets more votes;

- they replace the arcs between every pair of alternatives with an arc oriented from the winning to the losing alternative and labeled with the positive difference of the votes.

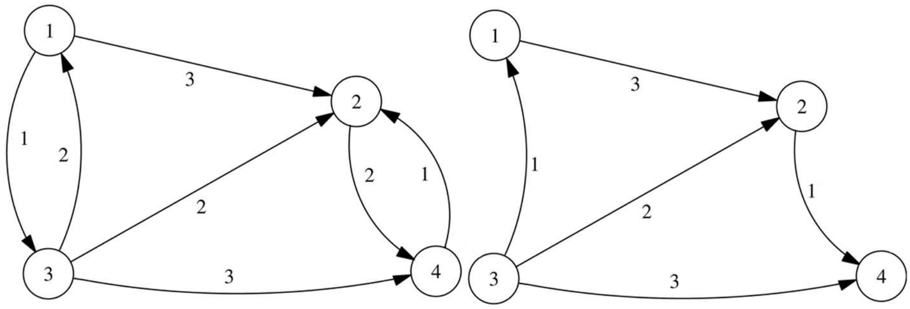

Figure 8: *A problematic multigraph with the associated reduced multigraph*

In the case of the multigraph on the left side of Figure 8 this procedure allows the definition of the following relations between each pair of alternatives: a_3 wins over a_1 for two votes against one (with a difference of one), a_1 wins over a_2 for three votes against none (with a difference of three), a_3 wins over a_2 for two votes against none (with a difference of two), a_3 wins over a_4 for three votes against none (with a difference of three) and a_2 wins over a_4 for two votes against one (with a difference of one). If such winning relations are translated in directed arcs between each pair of alternatives they allow the definition of a reduced multigraph in which, between each pair of alternatives, we have at the most one arc labeled with the foregoing differences and properly oriented.
As it is shown in the right side of Figure 8 we have that the application of this Condorcet-like procedure defines a reduced multigraph in which the only node without incoming arcs is node 3 so that the potentially best alternative is a_3 (or $\hat{A} = \{a_3\}$), as it is easily verified by using both *backward* and *forward* pruning on the reduced multigraph (in which the only node without outgoing arcs is node 4 that corresponds to a_4 so that $\check{A} = \{a_4\}$). This is not always the case as it is shown, as an extreme case, in Figure 9. For the multigraph of this figure the application of the Condorcet-like procedure would give us a reduced multigraph made only of isolated nodes since the alternative of each pair are tied in pairwise comparisons. In this case the deciders would not be able either to identify the best alternative or to declare the alternatives as equivalent owing to the differences in the multiplicity indexes among pairs of alternatives. This is the reason why we call this multigraph **unsolvable**.

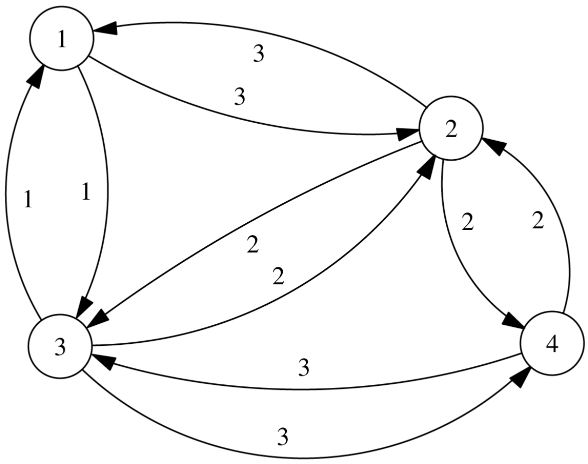

Figure 9: *An unsolvable multigraph*

If the *merawti* method produces an unsolvable multigraph we must admit that it failed.
The only possible way out is that the deciders, possibly after a revision of both the sets of the criteria and the sets of the alternatives, asses again the alternatives and repeat the construction of their graphs G_i so to produce a solvable MG to which possibly there corresponds a reduced graph with no isolated nodes.
We note, indeed, how in intermediate cases, in which the reduced multigraph has both isolated nodes and connected nodes, the *merawti* method allows the deciders to rank the alternatives corresponding to the connected nodes but without allowing them to attain the identification of a single best solution so we must admit, again, its (at least partial) failure although its usefulness as a decision aiding tools is still valid.

7 The final selection

In this section we assume that the multigraph MG has been defined by the deciders that are able to identify a non-empty set $\hat{A}$ of the best alternatives.
The use of both *backward* and *forward* pruning allows us to verify if an alternative is either the best one or the worst one if the corresponding sets contain only one element. In other cases, such as those that we show in Figure 10, these methods prove either useless or unsatisfactory.
In all these three cases we have, indeed, that there is no worst alternative (since there is no node without outgoing arcs but with incoming arcs and so we have $\check{A} = \emptyset$) whereas in the two cases (b) and (c) we have that the set $\hat{A}$ contains two alternatives.
From our definitions of both *backward* and *forward* pruning we see how the

three cases of Figure 10 are intractable with these tools. We indeed cannot apply backward pruning since we are not able to identify a starting worst alternative and, if we apply forward pruning, we end up with alternatives a_2 and a_4 connected in a cycle although with an indication that a_4 is believed to be worse than a_2 (since $2 > 1$).
The case (a) can be dealt with by considering that a_3 is the only alternative that is not less preferred than any other alternative so that it can be considered as the best alternative of the set A according to the criteria of the set C. In this case, indeed, we have $\hat{A} = \{a_3\}$.

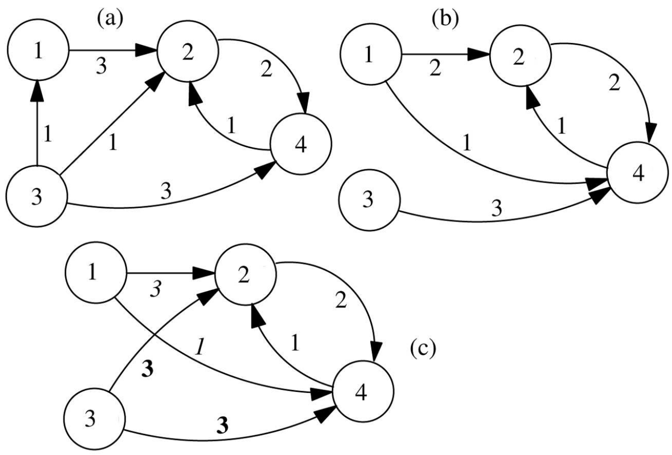

Figure 10: *Three examples of graphs for the final selection*

In order to handle the other two cases where $\hat{A} = \{a_1, a_3\}$ we propose a **lexicographic approach** that considers, for each node, the number of the outgoing nodes and the number of the outgoing arcs, counted with their multiplicity.
More formally we associate to each $a_i \in \hat{A}$ a pair of integer values (y_i, x_i) where x_i counts, with their multiplicity, the number of outgoing arcs from the node i of the MG whereas y_i counts the number of the outgoing nodes for the same node of MG. At this point we define a preference relation $\sqsupset$ for each pair of alternatives $a_i, a_j \in \hat{A}$ as follows: we say that $a_i \sqsupset a_j$ (or the former alternative is preferred to the latter) if we have $y_i > y_j$ or $y_i = y_j$ and $x_i > x_j$; we say that these alternatives are equivalent or tied if we have $y_i = y_j$ and $x_i = x_j$ otherwise we have $a_j \sqsupset a_i$. Once this lexicographic ordering has been applied to the elements of the set $\hat{A}$ we can select the best alternative at random if we get a subset of tied best alternatives.
At this point we can apply this method to the (b) and (c) cases of Figure 10.
In the (b) case, since we have $y_1 = 2 > 1 = y_3$, we get $a_1 \sqsupset a_3$. This result seems appropriate and sensible since it privileges the alternative that is directly preferred to the higher number of other alternatives.
In the (c) case, since we have $y_3 = 2 = y_1$ and $x_3 = 6 > 4 = x_1$ we have $a_3 \sqsupset a_1$.

This result seems appropriate and sensible since, under the parity of the other parameter, it privileges the alternative that is preferred to the other alternatives by the higher number of deciders (six versus four in this case).

8 The robustness of the method

Up to this point we have assumed that the deciders use the sets A and C as an exogenous common knowledge of their decision process.
Under this assumption we note how each of the deciders can even define his own G_i without using the binary relation $>$ with the only constraint that each of them produces an acyclic connected graph of the common alternatives by ranking them with the common set of the criteria.
In addition to this degree of freedom we may analyze how the proposed method behaves if we relax the foregoing common knowledge condition though this can imply some minor modifications to the proposed method that anyway maintains its applicability. For these reasons we claim that the *merawti* method is robust. With this we mean that each decider d_i can enter the decision process with his own set of alternatives A_i or his own set of criteria C_i or both so that these sets are termed as endogenously defined in the decision process itself. We can state that every set A_i represents the private knowledge of d_i whereas the set C_i is associated to his values and beliefs system.
The first way in which we can relax the common knowledge condition is the following. The deciders share the set A but each of them has his own set C_i. In this case the method can be applied as it is since each decider can produce his connected acyclic graph G_i that is merged together with those of the other deciders in a single multigraph MG. It is evident, however, how the conclusions that we can derive from the application of the method are the more convincing the higher is the number of the criteria that are shared by the deciders and so the higher is the cardinality of $\cap_i C_i$.
A second way is that the deciders share the set C but each of them has his own set A_i of the alternatives as a personal feasibility set. In this case the method must be modified so to handle these differences. The basic hypothesis is $A_c = \cap_i A_i \neq \emptyset$ otherwise the deciders have nothing in common about which to deliberate. The set A_c represents the common knowledge of the deciders in this case. Under this hypothesis the *merawti* method can be applied as it is on the set A_c so to produce the set $\hat{A}$ that we assume not empty. For a treatment of the empty set case we refer to section 6.
At this point one or more deciders can request for the evaluation also of the alternatives that belong to some A_i but not to A_c. If the proposal is unanimously refused by the non proposers or the request is not formulated the final selection is performed on the set $\hat{A}$.
If the proposal is accepted the execution of the method is repeated from scratch on the enlarged set composed by A_c and the proposed non common alternatives. The method produces, therefore, a new set $\hat{A}$ that can undergo the final selection unless there is some other accepted request that forces the deciders to

reapply the method from the start once again. We note that since $\cup_i A_i$ is finite and since every alternative that is not initially in A_c can be examined only once this iterative process cannot last forever. If, at any iteration, the method produces an empty set $\hat{A}$ we discard the corresponding iteration as useless together with the alternatives not in A_c that have been proposed at that iteration.
The third and last way is when every decider has his own set A_i of the alternatives and his own set C_i of the criteria. In this case the method is applied over the set A_c as we have seen in the first way but with the possible use of alternatives that do not belong to the set A_c as we have seen in the second way.

9 Conclusions and future plans

In this Technical Report we have presented a multideciders multicriteria method that can be used by d deciders as an aiding tool for the selection of the best alternatives from a set A of a alternatives according to the c criteria of the set C.
The method is simple and is composed of an individual step and a collective step that aim at producing a single oriented multigraph that the deciders can use to perform the final selection.
In the Technical Report we have shown the method at work together with some of its strengths and weaknesses.
Future plans include a deeper analysis of the properties of the proposed method together with their formalization. We also aim at characterizing the set of the criteria and at applying the method in more complex and more realistic cases.
Other streams of research that are surely worth of being pursued include:

- a deeper analysis of the failure cases that we have presented in section 6;
- a more formal analysis of the handling of the irrationality at which we only hinted in section 4;
- an analysis of the fragility of the method to strategic behaviors that we mentioned in section 1.

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