Università di Pisa
Dipartimento di Informatica

Technical Report

On Linearising Mixed-Integer Quadratic Programs via Bit Representation

Laura Galli Adam N. Letchford

June 2018

ADDRESS: Largo B. Pontecorvo 3, 56127 Pisa, Italy. TEL: +39 050 2212700 FAX: +39 050 2212726

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Abstract

It is well known that, under certain conditions, one can use bit representation to transform both integer quadratic programs and mixed-integer bilinear programs into mixed-integer linear programs (MILPs), and thereby render them easier to solve using standard software packages. We show how to convert a more general family of mixed-integer quadratic programs to MILPs, and present several families of strong valid linear inequalities that can be used to strengthen the continuous relaxations of the resulting MILPs.

Keywords: mixed-integer nonlinear programming, linearisation

1 Introduction

A wide range of problems in Operational Research, Statistics, Quantitative Finance and Engineering can be formulated as mixed-integer quadratic programs (MIQPs), i.e., optimisation problems with a mixture of continuous and integer-constrained variables, linear constraints, and a quadratic objective function. Although some sophisticated algorithms have been developed to solve MIQPs (see, e.g., [2, 11, 13, 14]), they can still present a formidable challenge, especially if the objective function is non-convex.

It has been known for some time that, under certain conditions, an MIQP can be converted into (or "reformulated as") a mixed-integer linear program (MILP), via the use of additional variables and constraints. This has been shown for 0-1 quadratic programs (e.g., [6, 7]), integer quadratic programs with bounded variables (e.g., [3, 15]), and various integer and mixed-integer bilinear programs (e.g., [7, 9, 10]). These results are of interest because a wide range of excellent software packages are now available for solving

*Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56124 Pisa, Italy. E-mail: laura.galli@unipi.it

†Department of Management Science, Lancaster University, Lancaster LA1 4YW, United Kingdom. E-mail: a.n.letchford@lancaster.ac.uk

MILPs to optimality. (Prominent examples include `CPLEX`, `Gurobi`, `LINDO`, `SCIP` and `Xpress`.)

The methods in [3, 7, 9, 10, 15] are based on bit representation. In this paper, we extend them to a broader family of MIQPs; namely, bounded MIQPs in which the objective function contains no products or squares of continuous variables. We also present several families of strong valid linear inequalities that can be used to strengthen the continuous relaxations of the resulting MILPs.

The paper has a very simple structure. The literature is reviewed in Section 2, the extensions are given in Section 3, and the valid inequalities are presented in Section 4. Throughout the paper, we assume that MIQPs are written in the following form:

$$\min\left\{x^T Q x + c \cdot x : \ Ax \le b, \ x \in \mathbb{R}^n_+, \ x_i \in \mathbb{Z} \ (i \in I)\right\},$$

where $Q \in \mathbb{Q}^{n \times n}$, $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $I \subseteq \{1, \ldots, n\}$. We also let N denote $\{1, \ldots, n\}$.

2 Literature Review

In this section, we review the relevant literature, in chronological order. We remark that, due to space restrictions, we have had to be rather selective.

In 1959, Fortet [6] showed how to linearise 0-1 quadratic programs (0-1 QPs). We replace each quadratic term, say $x_i x_j$, with a new binary variable, say X_{ij}, and add the constraints

$$X_{ij} \ge 0, \ X_{ij} \le x_i, \ X_{ij} \le x_j, \ x_i + x_j - X_{ij} \le 1. \tag{1}$$

This leads to a 0-1 linear program (0-1 LP).

In 1967, Watters [15] considered the more general case of integer quadratic programs (IQPs) with bounded variables. Suppose we know that $x_i \le u_i$, where u_i is a positive integer. Let r_i denote $\lfloor \log_2 u_i \rfloor$, and replace x_i with

$$\sum_{s=0}^{\lfloor \log_2 u_i \rfloor} 2^s \, \tilde{x}_{is},$$

where the $\tilde{x}_{is}$ are new binary variables. If $u_i + 1$ is not a power of two, one must also add the constraint:

$$\sum_{s=0}^{\lfloor \log_2 u_i \rfloor} 2^s \, \tilde{x}_{is} \le u_i.$$

In this way, we convert the IQP into a 0-1 QP. One can then apply the method of Fortet to convert the 0-1 QP into a 0-1 LP.

In 1975, Glover [7] proposed a more parsimonious way to tackle 0-1 QPs. For $i \in N$, define a new variable, say λ_i, representing the quantity

$$x_i \sum_{j \in N} Q_{ij} x_j,$$

and replace the objective function with $\sum_{i \in N} \lambda_i$. Then, to link together the x and λ variables, add the following constraints for $i \in N$:

$$L_i x_i \le \lambda_i \le U_i x_i \tag{2}$$

$$\sum_{j=1}^{n} Q_{ij} x_j - U_i(1 - x_i) \le \lambda_i \le \sum_{j=1}^{n} Q_{ij} x_j - L_i(1 - x_i), \tag{3}$$

where L_i and U_i are lower and upper bounds on $\sum_{j \in N} Q_{ij} x_j$. (Suitable values for L_i and U_i are $\sum_{j \in N} \min \{0, Q_{ij}\}$ and $\sum_{j \in N} \max \{0, Q_{ij}\}$, respectively.) The result is a mixed 0-1 LP.

In the same paper, Glover considered bounded mixed-integer bilinear programs (MIBPs) in which, in each bilinear term, at least one of the variables is integer. He proposed the following approach. First, each integer variable x_i is replaced with binary variables $\tilde{x}_{is}$, *à la* Watters. Second, for all $i \in I$ and all s, a variable y_{is} is introduced, representing

$$\tilde{x}_{is} \sum_{j \in N} Q_{ij} x_j.$$

These variables are used to linearise the objective. Finally, constraints similar to (2) and (3) are used to link the y_{is} variables with the $\tilde{x}_{is}$ variables and any remaining x_i variables. The result is again a mixed 0-1 LP.

Adams & Sherali [1] showed that one can strengthen the LP relaxations of Fortet-type formulations as follows. Take any linear constraint from the original 0-1 QP, say $\alpha \cdot x \le \beta$, and any $i \in N$, and note that the quadratic inequalities $(\alpha \cdot x) x_i \le \beta x_i$ and $(\alpha \cdot x)(1 - x_i) \le \beta(1 - x_i)$ are valid. Linearising them yields $2n$ inequalities of the form:

$$\sum_{j \ne i} \alpha_j X_{ij} \le (\beta - \alpha_i) x_i$$

$$\sum_{j \ne i} \alpha_j (x_j - X_{ij}) \le \beta(1 - x_i).$$

This approach is now called the *Reformulation-Linearization Technique* (RLT), and the inequalities are called *RLT* inequalities [13].

In 1989, Padberg [12] studied the so-called *Boolean quadric polytope*, which is the convex hull of pairs $(x, X) \in \{0, 1\}^{n + \binom{n}{2}}$ satisfying (1). He derived several families of cutting planes for that polytope, that can be used to further strengthen Fortet relaxations. (For a survey of additional known cutting planes, see Section V of [5].)

In 1997, Harjunkoski *et al.* [10] gave a new approach for bounded integer bilinear programs (IBPs). First, each integer variable x_i is replaced with binary variables $\tilde{x}_{is}$, as usual. Then, for all pairs $i, j \in N$, and for $s = 0, \ldots, r_i$, an additional continuous variable, say v_{isj}, is defined, which represents the product $\tilde{x}_{is} x_j$. They then replace all terms of the form $x_i x_j$ with $\sum_{s=0}^{r_i} 2^s v_{isj}$. Finally, they add the following linear inequalities for all pairs i, j and for $s = 0, \ldots, r_i$:

$$v_{isj} \geq 0, \; v_{isj} \leq u_j \tilde{x}_{is}, \; v_{isj} \leq x_j, \; v_{isj} \geq u_j \tilde{x}_{is} + x_j - u_j. \tag{4}$$

The result is again a mixed 0-1 LP. We remark that this approach can be easily extended to bounded IQPs, just by allowing i and j to be identical.

In 2008, Billionnet *et al.* [3] rediscovered the approach in [10], in the context of IQPs. They also used the RLT to derive cutting planes for the resulting mixed 0-1 LPs. In addition, they noted that the following equations are valid for all $\{i, j\} \subseteq N$:

$$\sum_{s=0}^{r_i} 2^s \, v_{isj} \; = \; \sum_{s=0}^{r_j} 2^s \, v_{jsi}. \tag{5}$$

In 2012, Günluk *et al.* [8] found a "hybrid" method for bounded IBPs, involving a combination of the $\tilde{x}_{is}$ and X_{ij} variables. The resulting 0-1 LP has an exponential number of constraints, but they present an efficient separation algorithm for those constraints.

Finally, in 2013, Gupte *et al.* [9] adapted the method in [10] to the case of bounded MIBPs in which each bilinear term is the product of an integer variable and a continuous variable. They also derived some cutting planes, as follows. For a given $i \in I$, let S_i^0 and S_i^1 be the sets of bits that take the value zero or one, respectively, in the bit representation of u_i. For any $s \in S_i^0$, if we let $C(s)$ denote $\{t \in S_i^1 : t > s\}$, then the linear inequality

$$\sum_{t \in C(s) \cup \{s\}} \tilde{x}_{it} \leq |C(s)| \tag{6}$$

is valid. Following the RLT, we can multiply each such inequality by either x_j or $u_j - x_j$, for any $j \in N \setminus I$, to obtain the inequalities

$$\sum_{t \in C(s) \cup \{s\}} v_{itj} \leq |C(s)| x_j \tag{7}$$

$$u_j \sum_{t \in C(s) \cup \{s\}} \tilde{x}_{it} - \sum_{t \in C(s) \cup \{s\}} v_{itj} \leq |C(s)|(u_j - x_j). \tag{8}$$

3 Linearisations for a Broad Family of MIQPs

In this section, we extend the results in [6, 7, 10, 15] to cover a more general family of bounded MIQPs. We will need the following definition.

Definition 1 *A bounded MIQP is "nice" if the quadratic term $x^T Qx$ contains no products or squares of continuous variables.*

Note that nice MIQPs include 0-1 QPs, bounded IBPs and bounded IQPs as special cases, as well as the MIBPs considered by Glover [7] and Gupte *et al.* [9]. In the following three subsections, we present three strategies for converting nice MIQPs into mixed 0-1 LPs.

3.1 Strategy G

The first strategy, which we call "Strategy G", is an extension of the one in Glover [7]. It involves the following steps.

1. For $i \in I$, replace x_i with $\sum_{s=0}^{r_i} 2^s \tilde{x}_{is}$ in the objective and constraints.

2. For $i \in I$ and $s = 0, \ldots, r_i$, introduce a continuous variable, say y_{is}, representing the quantity $\tilde{x}_{is} \sum_{j \in N} Q_{ij} x_j$.

3. In the objective function, replace the term $x^T Qx$ with
$$\sum_{i \in I} \sum_{s=0}^{r_i} 2^s \, y_{is}.$$

4. For $i \in I$, compute lower and upper bounds, say L_i and U_i, on the value that can be taken by $\sum_{j \in N} Q_{ij} x_j$ in any feasible solution. (Suitable values for L_i and U_i are $\sum_{j \in N} \min \{0, Q_{ij}\} u_j$ and $\sum_{j \in N} \max \{0, Q_{ij}\} u_j$, respectively.)

5. Add the following constraints for $i \in I$ and $s = 0, \ldots, r_i$:
$$L_i \tilde{x}_{is} \le y_{is} \le U_i \tilde{x}_{is}$$
$$y_{is} \ge \textstyle\sum_{j \in N \setminus I} Q_{ij} x_j + \sum_{j \in I} Q_{ij} \sum_{t=0}^{r_j} 2^t \tilde{x}_{jt} - U_i(1 - \tilde{x}_{is})$$
$$y_{is} \le \textstyle\sum_{j \in N \setminus I} Q_{ij} x_j + \sum_{j \in I} Q_{ij} \sum_{t=0}^{r_j} 2^t \tilde{x}_{jt} - L_i(1 - \tilde{x}_{is}).$$
One can check that these constraints force y_{is} to equal zero when $\tilde{x}_{is}$ is zero, and to equal $\tilde{x}_{is} \sum_{j \in N} Q_{ij} x_j$ when $\tilde{x}_{is}$ is one.

The resulting mixed 0-1 LP has only $O(n + L)$ variables and $O(m + n + L)$ constraints, where L denotes $\sum_{i \in I} (r_i + 1)$.

3.2 Strategy H

The second strategy, which we call "Strategy H", is an extension of the one in Harjunkoski *et al.* [10].

1. For $i \in I$, replace x_i with $\sum_{s=0}^{r_i} 2^s \tilde{x}_{is}$ in the objective and constraints.

2. For $i \in I$, $s = 0, \ldots, r_i$ and $j \in N$, introduce a continuous variable v_{isj}, representing the product $\tilde{x}_{is} x_j$. (We permit $j = i$ here, unlike in the bilinear case.)

3. For $i \in I$ and $j \in N$, replace $x_i x_j$ with $\sum_{s=0}^{r_i} 2^s v_{isj}$.

4. For $i \in I$, $s = 0, \ldots, r_i$ and $j \in N \setminus I$, add the constraints (4).

5. For $i, j \in I$, not necessarily distinct, and $s = 0, \ldots, r_i$, add the constraints

$$v_{isj} \geq 0,\ v_{isj} \leq u_j \tilde{x}_{is},\ v_{isj} \leq \sum_{t=0}^{r_j} 2^t \tilde{x}_{jt},\ v_{isj} \geq u_j \tilde{x}_{is} + \sum_{t=0}^{r_j} 2^t \tilde{x}_{jt} - u_j. \tag{9}$$

The resulting mixed 0-1 LP has $O(nL)$ variables and $O(m+nL)$ constraints.

3.3 Strategy FH

The third and final strategy, which we call "Strategy FH", is a kind of "hybrid" of the ones in Fortet [6] and Harjunkoski *et al.* [10]. (It also generalises the approaches in [3, 9].)

1. For $i \in I$, replace x_i with $\sum_{s=0}^{r_i} 2^s \tilde{x}_{is}$ in the objective and constraints.

2. For $i \in I$, $s = 0, \ldots, r_i$ and $j \in N \setminus I$, introduce a continuous variable v_{isj}, representing $\tilde{x}_{is}\, x_j$.

3. For all $\{i, j\} \subseteq I$ with $i < j$, $s = 0, \ldots, r_i$ and $t = 0, \ldots, r_j$, introduce a new binary variable $\tilde{X}_{isjt}$, representing $\tilde{x}_{is}\, \tilde{x}_{jt}$.

4. For $i \in I$ and $0 \leq s < t \leq r_i$, introduce a binary variable $\tilde{X}_{isit}$, representing $\tilde{x}_{is}\, \tilde{x}_{it}$.

5. Use the v and $\tilde{X}$ variables to linearise the objective function.

6. For $i \in I$, $j \in N \setminus I$ and $s = 0, \ldots, r_i$, add the constraints (4).

7. For all $\{i, j\} \subseteq I$ with $i < j$, $s = 0, \ldots, r_i$ and $t = 0, \ldots, r_j$, add the Fortet-type constraints:

$$\tilde{X}_{ijst} \leq \tilde{x}_{is},\ \tilde{X}_{ijst} \leq \tilde{x}_{jt},\ \tilde{x}_{is} + \tilde{x}_{jt} - \tilde{X}_{ijst} \leq 1.$$

8. For $i \in I$ and $0 \leq s < t \leq r_i$, add the Fortet-type constraints

$$\tilde{X}_{isit} \leq \tilde{x}_{is},\ \tilde{X}_{isit} \leq \tilde{x}_{it},\ \tilde{x}_{is} + \tilde{x}_{it} - \tilde{X}_{isit} \leq 1.$$

The resulting mixed 0-1 LP has $\mathcal{O}\big(L(n+L)\big)$ variables and $\mathcal{O}\big(m+L(n+L)\big)$ constraints.

We close this section with a couple of remarks.

Remark 1 *Strategy H can be regarded as a "disaggregation" of Strategy G, in the sense that each λ_{is} variable can be expressed as a linear combination of v_{isj} variables. Similarly, Strategy FH can be regarded as a disaggregation of Strategy H, in the sense that, when both i and j are in I, each v_{isj} variable can be expressed as a linear combination of $\tilde{X}_{isjt}$ variables.*

Remark 2 *Recently, Del Pia* et al. *[4] showed that, given an arbitrary (not necessarily bounded) MIQP, there exists at least one rational optimal solution that can be encoded using a number of bits that is bounded by a polynomial of the input size. From this it follows that any MIQP whose objective function does not include products or squares of continuous variables can be reformulated as a mixed 0-1 LP of polynomial size.*

4 Valid Inequalities

In this section, we present some valid inequalities that can be used to improve the continuous relaxations of the mixed 0-1 LPs that one obtains via Strategies H and FH.

4.1 Valid inequalities for Strategy H

Consider the mixed 0-1 LP that arises when one uses Strategy H. Recall that it contains continuous variables x_i for $i \in N \setminus I$, binary variables $\tilde{x}_{is}$ for $i \in I$ and $s = 0, \ldots, r_i$, and continuous variables v_{isj} for $i \in I$ and $s = 0, \ldots, r_i$.

First, we can adapt the inequalities of Billionnet *et al.* [3] to our setting. The steps are as follows.

- For $i \in I$, $s = 0, \ldots, r_i$ and $k = 1, \ldots, m$, take the kth linear constraint in the system $Ax \leq b$, and multiply it by either $\tilde{x}_{is}$ or $1 - \tilde{x}_{is}$, to yield a quadratic inequality.
- For all $i \in I$ such that u_i+1 is not a power of two, and for $k = 1, \ldots, m$, take the kth linear constraint, and multiply it by $u_i - x_i$ to yield another quadratic inequality.
- Linearise the resulting quadratic inequalities, by expressing them in terms of the x, $\tilde{x}$ and v variables.

For example, multiplying a linear constraint of the form $\alpha \cdot x \leq \beta$ by $\tilde{x}_{is}$, and linearising, yields

$$\sum_{j \in N} \alpha_j \sum_{s=0}^{r_i} 2^s \, v_{isj} \ \leq \ \beta \tilde{x}_{is}.$$

Next, we note that the equations (5) are also valid for all $\{i,j\} \subseteq I$. Moreover, following Gupte *et al.* [9], we can add the inequalities (7) and (8) for all $i \in I$, all $s \in S_i^0$ and all $j \in N \setminus I$.

The inequalities mentioned so far can all be derived via the RLT. The following proposition shows that two other families of inequalities can be derived via the RLT.

Proposition 1 *We can derive additional valid inequalities in $\big(x, \tilde{x}, v\big)$-space as follows.*

- *For all (not necessarily distinct) pairs $i, j \in I$, and for all $s \in S_i^0$, multiply the inequality (6) by either x_j or $u_j - x_j$, and linearise.*
- *For $i \in I$, $s \in S_i^0$, and $k = 1, \ldots, m$, multiply the inequality (6) by the kth linear inequality in the system $Ax \le b$, and linearise.*

For example, multiplying (6) by $u_j - x_j$ and linearising yields

$$u_j \sum_{t \in C(s) \cup \{s\}} \tilde{x}_{it} - \sum_{t \in C(s) \cup \{s\}} v_{itj} \ \le \ |C(s)| \left(u_j - \sum_{t=0}^{r_j} 2^t \tilde{x}_{jt} \right). \tag{10}$$

We remark that, if the system $Ax \le b$ contains any constraints that involve only integer variables, one can generate still more valid inequalities via the RLT, by multiplying such constraints by either x_i or $u_i - x_i$ for some $i \in N \setminus I$.

The following two propositions present some additional inequalities that are *not* derived via the RLT.

Proposition 2 *When $i = j$, we can strengthen the constraints (9) as follows:*

- *Replace the first inequality in (9) with*

$$v_{isi} \ge 2^s \, \tilde{x}_{is}. \tag{11}$$

- *Replace the second inequality in (9) with*

$$v_{isi} \le \lambda_{is}^1 \, \tilde{x}_{is}, \tag{12}$$

  where λ_{is}^1 is the largest value that x_i can take when $\tilde{x}_{is} = 1$.

- *Replace the fourth inequality in (9) with*

$$v_{isi} \ge \sum_{s=0}^{r_i} 2^s \tilde{x}_{is} + \lambda_{is}^0 (\tilde{x}_{is} - 1), \tag{13}$$

  where λ_{is}^0 is the largest value that x_i can take when $\tilde{x}_{is} = 0$.

Proof. If $\tilde{x}_{is} = 0$, then $v_{isi} = 0$. If $\tilde{x}_{is} = 1$, then both x_i and v_{isi} must lie between 2^s and λ^1_{is}. Either way, the inequalities (11) and (12) are satisfied. As for the inequality (13), its right-hand side is equivalent to x_i when $\tilde{x}_{is} = 1$, and to $x_i - \lambda^0_{is}$ when $\tilde{x}_{is} = 0$. In either case, the inequality is satisfied. $\square$

Proposition 3 *When $i = j$, the inequalities (10) can be strengthened to:*

$$\tilde{\lambda}_{is} \sum_{t \in \{s\} \cup C(s)} \tilde{x}_{it} - \sum_{t \in \{s\} \cup C(s)} v_{iti} \le |C(s)| \left(\tilde{\lambda}_{is} - \sum_{t=0}^{r_j} 2^t \tilde{x}_{it} \right), \tag{14}$$

where $\tilde{\lambda}_{is}$ is the largest value that x_i can take when at least two of the bits in $\{s\} \cup C(s)$ must take the value zero.

Proof. Observe that x cannot take a value larger than $\tilde{\lambda}_{is}$ if the inequality (6) has a positive slack. This implies the following quadratic inequality:

$$\left(|C(s)| - \sum_{t \in C(s) \cup \{s\}} \tilde{x}_{it} \right) \left(\tilde{\lambda}_{is} - x \right) \ge 0.$$

(To see this, note that the first quantity is always non-negative, and the second term can only be negative if the first is zero.) Expanding the quadratic inequality and linearising yields (14). $\square$

4.2 Valid inequalities for Strategy FH

Now consider the mixed 0-1 LP that arises when one uses Strategy FH. Recall that it contains continuous variables x_i for $i \in N \setminus I$, binary variables $\tilde{x}_{is}$ for $i \in I$ and $s = 0, \ldots, r_i$, continuous variables v_{isj} for $i \in I$, $s = 0, \ldots, r_i$ and $j \in N \setminus I$, and binary variables $\tilde{X}_{isjt}$ for $i, j \in I$, $s = 0, \ldots, r_i$ and $t = 0, \ldots, r_j$.

A first observation is that, as in the case of Strategy H, we can add the inequalities (7) and (8) for all $i \in I$, all $s \in S^0_i$ and all $j \in N \setminus I$.

A second observation is that we can again derive several families of inequalities using the RLT. Details are given in the following proposition.

Proposition 4 *We can derive additional valid inequalities in $\left(x, \tilde{x}, v, \tilde{X}\right)$-space as follows.*

- *For $i \in I$, $s = 0, \ldots, r_i$, and $k = 1, \ldots, m$, multiply the kth linear inequality in the system $Ax \le b$ by either $\tilde{x}_{is}$ or $1 - \tilde{x}_{is}$, and linearise.*
- *For all (not necessarily distinct) pairs $i, j \in I$, $s = 0, \ldots, r_i$ and $t = 0, \ldots, r_j$, multiply the inequality (6) by either $\tilde{x}_{jt}$ or $1 - \tilde{x}_{jt}$, and linearise.*

- *For $i \in I$, $s \in S_i^0$, and $k = 1, \ldots, m$, multiply the inequality (6) by the kth linear inequality in the system $Ax \leq b$, and linearise.*
- *For all (not necessarily distinct) pairs $i, j \in I$, $s \in S_i^0$ and $t \in S_j^0$, multiply the inequality (6) by the analogous inequality $\sum_{\ell \in C(t) \cup \{t\}} \tilde{x}_{j\ell} \leq |C(t)|$, and linearise.*

For example, if we apply the last operation mentioned, we obtain

$$|C(s)| \sum_{\ell \in C(t) \cup \{t\}} \tilde{x}_{j\ell} + |C(t)| \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \leq |C(s)||C(t)| + \sum_{k \in C(s) \cup \{s\}} \sum_{\ell \in C(t) \cup \{t\}} \tilde{X}_{ikj\ell}. \tag{15}$$

As in the previous subsection, if the system $Ax \leq b$ contains any constraints that involve only integer variables, even more valid inequalities can be generated via the RLT.

Our last proposition shows how to strengthen (15) in some cases.

Proposition 5 *When $i = j$ and $s = t$, inequality (15) can be strengthened to:*

$$\big(|C(s)| - 1\big) \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \leq \binom{|C(s)|}{2} + \sum_{\{k,\ell\} \subseteq C(s) \cup \{s\}} \tilde{X}_{iki\ell}. \tag{16}$$

Proof. Since $p(p-1) \geq 0$ for all integers p, the quadratic inequality

$$\left(|C(s)| - \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \right) \left(|C(s)| - \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} - 1 \right) \geq 0$$

is valid. Expanding yields:

$$\big(2|C(s)| - 1\big) \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \leq |C(s)|\big(|C(s)| - 1\big) + \left(\sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \right)^2.$$

Using the identities $\tilde{x}_{ik}^2 = \tilde{x}_{ik}$ and $\tilde{x}_{ik}\tilde{x}_{i\ell} = \tilde{X}_{iki\ell}$, we obtain:

$$\big(2|C(s)| - 2\big) \sum_{k \in C(s) \cup \{s\}} \tilde{x}_{ik} \leq |C(s)|\big(|C(s)| - 1\big) + 2 \sum_{\{k,\ell\} \subseteq C(s) \cup \{s\}} \tilde{X}_{iki\ell}.$$

Dividing by two yields (16). □

To conclude, we mention that a natural topic for future research is to perform a thorough computational comparison of the various formulations and cutting planes proposed in this paper, with view to finding out which tend to be most useful in practice.

Acknowledgement: The authors received some financial support from the Università di Pisa, under the project "Modelli ed algoritmi innovativi per problemi strutturati e sparsi di grandi dimensioni" (grant PRA_2017_05).

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