Università di Pisa
Dipartimento di Informatica

Technical Report

Bit Representation Can Improve SDP Relaxations of Mixed-Integer Quadratic Programs

Laura Galli Daniel J. Grainger Adam N. Letchford

July 2018

ADDRESS: Largo B. Pontecorvo 3, 56127 Pisa, Italy. TEL: +39 050 2212700 FAX: +39 050 2212726

Bit Representation Can Improve SDP Relaxations of Mixed-Integer Quadratic Programs

Laura Galli* Daniel J. Grainger† Adam N. Letchford‡

July 2018

Abstract

A standard trick in integer programming is to replace bounded integer variables with binary variables, using a bit representation. In a previous paper, we showed that this process can be used to improve linear programming relaxations of mixed-integer quadratic programs. In this paper, we show that it can also be used to improve *semidefinite* programming relaxations.

Keywords: mixed-integer nonlinear programming, global optimisation, semidefinite programming.

1 Introduction

A folklore result in integer programming is that bounded integer-constrained variables can be replaced with binary variables. In particular, if x_i is known to lie in $[0, u_i] \cap \mathbb{Z}$, where $u_i \geq 2$ and integer, then one can replace x_i with its *bit representation*

$$\sum_{s=0}^{\lfloor \log_2 u_i \rfloor} 2^s \, \tilde{x}_{is},$$

where the $\tilde{x}_{is}$ are new binary variables [33].

In [4, 25, 26, 29], bit representation was used to derive strong *linear programming* (LP) relaxations of bounded mixed-integer linear programs. It has also been used to derive strong LP and *quadratic programming* (QP) relaxations of bounded non-convex *mixed-integer quadratic programs* (MIQPs) [3, 14, 16]. Our goal in this paper is to show that it can also be used to derive strong *semidefinite programming* (SDP) relaxations of MIQPs. Perhaps surprisingly, this is true even in the convex case.

*Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56124 Pisa, Italy. E-mail: `Laura.Galli@di.unipi.it`

†Former PhD student at Lancaster University.

‡Department of Management Science, Lancaster University, Lancaster LA1 4YW, United Kingdom. E-mail: `A.N.Letchford@lancaster.ac.uk`

The paper has a very simple structure. The relevant literature is reviewed in Section 2, and the new results are presented in Section 3.

Throughout the paper, we assume that we are given a bounded MIQP with n variables and m constraints, in the form:

$$\min \quad x^T Q x + c \cdot x \tag{1}$$
$$Ax \leq b \tag{2}$$
$$x_i \in [0, u_i] \qquad (i \in N) \tag{3}$$
$$x_i \in \mathbb{Z} \qquad (i \in I), \tag{4}$$

where $Q \in \mathbb{Q}^{n \times n}$, $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $N = \{1, \ldots, n\}$, $I \subseteq N$, and $u_i > 0$ for all $i \in N$. We assume without loss of generality that Q is symmetric and that the u_i are positive integers. We also let r_i denote $\lfloor \log_2 u_i \rfloor$ for all $i \in I$. Finally, we let $\tilde{n}$ denote $n - |I| + \sum_{i \in I}(r_i + 1)$, which is the total number of variables obtained when the MIQP is converted into a mixed 0-1 QP, using bit representation.

2 Literature Review

For brevity, we review only SDP relaxations of quadratic problems in this section. For surveys of LP relaxations of quadratic problems, see, e.g., [7, 9, 14]. For surveys of SDP relaxations of 0-1 LPs, see, e.g., [20, 21].

2.1 SDP relaxation of 0-1 QPs

We begin with the special case of 0-1 QPs, i.e., problems of the form:

$$\min \left\{ x^T Q x : \; Ax \leq b, \, x \in \{0,1\}^n \right\}. \tag{5}$$

The standard SDP relaxation has its roots in early work of Lovász [23], Shor [31] and Körner [19]. We follow the presentation given in [22, 24, 27]. Introduce the matrix variable $X = xx^T$, along with the augmented matrix

$$X^+ = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Note that X^+ is psd by definition. Moreover, the condition $x_i \in \{0,1\}$ is equivalent to the (non-convex) quadratic equation $x_i^2 = x_i$. Accordingly, we can impose $X_{ii} = x_i$ for all i, or, equivalently, $\text{diag}(X) = x$. This leads to the following SDP relaxation of the 0-1 QP (5):

$$\min \left\{ Q \bullet X : \; Ax \leq b, \; \text{diag}(X) = x, \; X^+ \in \mathcal{S}_+^{n+1} \right\},$$

where '$\bullet$' denotes inner product and $\mathcal{S}_+^{n+1}$ denotes the cone of (real) psd matrices of order $n+1$. This basic relaxation can be strengthened via the addition of various kinds of cutting planes; see, e.g., [1, 17, 18, 24, 32]. As a simple example, one can add the constraints

$$X_{ij} \geq 0,\ X_{ij} \leq x_i,\ X_{ij} \leq x_j,\ X_{ij} \geq x_i + x_j - 1 \tag{6}$$

for $1 \leq i < j \leq n$ (e.g., [1, 18, 24]).

2.2 SDP relaxation of nonconvex QPs

Now consider a non-convex QP of the form:

$$\min \left\{ x^T Q x + c \cdot x : \ Ax \leq b,\ x \in \mathbb{R}^n_+ \right\}.$$

Constructing the augmented matrix X^+, as before, we obtain the following SDP relaxation [13, 28, 31]:

$$\inf \left\{ Q \bullet X + c \cdot x : \ Ax \leq b,\ X^+ \in \mathcal{S}^{n+1}_+ \right\}.$$

As noted by Anstreicher [2], however, this SDP is unbounded in general. To make it bounded, one must suppose that $x \leq u$ for some known $u \in \mathbb{Q}^n_+$. One can then enforce boundedness by adding the constraints

$$X_{ii} \leq u_i x_i \qquad (i \in N). \tag{7}$$

Anstreicher also notes one can strengthen the relaxation further by adding

$$X_{ij} \geq 0,\ X_{ij} \leq u_j x_i,\ X_{ij} \leq u_i x_j,\ X_{ij} \geq u_j x_i + u_i x_j - u_i u_j \tag{8}$$

for $1 \leq i < j \leq n$. Note that these constraints generalise (6). Additional cutting planes can be found in, e.g., [2, 6, 7, 12, 24, 30].

2.3 SDP relaxation of nonconvex MIQPs

Finally, we consider the case of general bounded MIQPs of the form (1)–(4). Given the constructions mentioned in the previous two subsections, a natural SDP relaxation is

$$\inf \left\{ Q \bullet X + c \cdot x : \ Ax \leq b,\ (7),\ X^+ \in \mathcal{S}^{n+1}_+ \right\}.$$

One can add the constraints (8) to strengthen this relaxation. Other valid inequalities can be found in, e.g., [5, 7, 8, 9, 15].

Billionnet *et al.* [3] suggest a rather different approach to bounded MIQPs. First, they convert the MIQP into a mixed 0-1 QP, via bit representation. Second, they show that, under certain technical assumptions, the mixed 0-1 QP can be convexified. Finally, they solve the convexified problem via branch-and-bound, with convex QP relaxations.

The motivation for applying bit representation in [3] was to convert the MIQP into a convex problem. We will show that bit representation has a happy side-effect: it can make the SDP relaxation stronger.

3 The Effect of Bit Representation

We now examine the effect of bit representation on the quality of lower bounds from SDP relaxations. In Subsection 3.1, we show that, if applied intelligently, it will never make the bounds worse. In Subsection 3.2, we show that it can make the bounds better, and explain why.

3.1 Bit representation never makes the bound worse

A major issue that could affect our analysis is that there are several different SDP relaxations of MIQP, which can be obtained by either including or omitting various families of cutting planes. To get around this, the following theorem deals with a "generic" SDP relaxation.

Theorem 1 *Consider a bounded MIQP of the form (1)–(4), and suppose that we have constructed an SDP relaxation of it (possibly with various cutting planes added). Let us write the SDP in the form:*

$$\begin{aligned} \inf \quad & Q^0 \bullet X + c^0 \cdot x \\ s.t. \quad & Q^j \bullet X + c^j \cdot x \leq b_j \quad (j = 1, \ldots, m) \\ & X^+ = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \end{aligned}$$

where $c^0, \ldots, c^m \in \mathbb{Q}^n$, $Q^0, \ldots Q^m \in \mathbb{Q}^{n \times n}$ and $b_1, \ldots, b_m \in \mathbb{Q}$. Now suppose that we apply the following three-step procedure. The first step is to replace:

- *x_i with $\sum_{s=0}^{r_i} 2^s \tilde{x}_{is}$ for $i \in I$,*
- *X_{ii} with $\sum_{s=0}^{r_i} \sum_{t=0}^{r_i} 2^{s+t} \tilde{X}_{isit}$ for $i \in I$,*
- *X_{ij} with $\sum_{s=0}^{r_i} \sum_{t=0}^{r_j} 2^{s+t} \tilde{X}_{isjt}$ for $\{i,j\} \subseteq I$,*
- *X_{ij} with $\sum_{s=0}^{r_i} 2^s \tilde{X}_{isj}$ for $i \in I$ and $j \in N \setminus I$.*

The second step is to add the constraints

$$\tilde{X}_{isis} = \tilde{x}_{is} \qquad (i = 1, \ldots, q; \ s = 1, \ldots, r_i).$$

In the third step, instead of imposing psd-ness on the augmented matrix X^+, we impose it on the expanded augmented matrix

$$\tilde{X}^+ = \begin{pmatrix} 1 & \tilde{x}^T \\ \tilde{x} & \tilde{X} \end{pmatrix},$$

where $\tilde{x}$ is the vector obtained from x by replacing each component x_i, for $i \in I$, with the components $\tilde{x}_{i,0}, \ldots, \tilde{x}_{i,r_i}$, and $\tilde{X}$ is a matrix variable intended to represent $\tilde{x}\tilde{x}^T$. (Note that $\tilde{x} \in \mathbb{R}^{\tilde{n}}$ and $\tilde{X} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, where $\tilde{n}$ is as defined at the end of Section 1.)

Then, the lower bound from the new SDP relaxation is no smaller than the one from the original relaxation.

Proof. It suffices to show that, given any feasible solution of the expanded SDP, there exists a feasible solution of the original SDP of the same value. To this end, let $(\tilde{x}^*, \tilde{X}^*) \in \mathbb{R}^{\tilde{n}+(\tilde{n})^2}$ be a feasible solution to the expanded SDP, let $(\tilde{X}^+)^* \in \mathbb{R}^{(\tilde{n}+1)^2}$ be the corresponding expanded augmented matrix, let $(x^*, X^*) \in \mathbb{R}^{n+n^2}$ be the corresponding pair defined by the mapping above, and let $(X^+)^* \in \mathbb{R}^{(n+1)^2}$ be the corresponding augmented matrix. By construction, (x^*, X^*) satisfies all of the constraints in the original SDP, and has the same cost as $(\tilde{x}^*, \tilde{X}^*)$. It therefore only remains to be shown that $(X^+)^*$ is psd. This is equivalent to showing that $v^T (X^+)^* v \geq 0$ for all vectors $v \in \mathbb{R}^{n+1}$. So, let $v = (v_0, \ldots, v_n)^T$ be such a vector and construct an expanded vector $\tilde{v}$ as follows. For $i \in I$, we replace the component v_i with the following $r_i + 1$ components:

$$v_i,\ 2v_i,\ \ldots,\ 2^{r_i} v_i.$$

Now, since $(\tilde{X}^+)^*$ is psd by assumption, we have $\tilde{v}^T (\tilde{X}^+)^* \tilde{v} \geq 0$. But $v^T (X^+)^* v = \tilde{v}^T (\tilde{X}^+)^* \tilde{v}$ by construction. □

3.2 Bit representation can make the bound better

The following example shows that bit representation can lead to an improvement in the lower bound from the SDP relaxation, even when (a) the objective function is convex, (b) there is only one variable, and (c) there are no constraints (apart from trivial bounds).

Example: Consider the trivial IQP

$$\min \left\{ x_1^2 - 3x_1 :\ x_1 \in \mathbb{Z} \cap [0,3] \right\}.$$

Following Anstreicher [2], the standard SDP relaxation of this IQP is

$$\min \left\{ X_{11} - 3x_1 :\ x_1 \in [0,3],\ X_{11} \leq 3x_1,\ X^+ = \begin{pmatrix} 1 & x_1 \\ x_1 & X_{11} \end{pmatrix} \succeq 0 \right\}.$$

One can check (either by hand or with an SDP solver) that the unique optimal solution to this SDP has $x_1 = 3/2$ and $X_{11} = 9/4$, giving a lower bound of $-9/4$. On the other hand, if we convert the IQP into a 0-1 QP, via bit representation, we obtain

$$\min \left\{ \tilde{x}_{10}^2 + 4\tilde{x}_{10}\tilde{x}_{11} + 4\tilde{x}_{11}^2 - 3\tilde{x}_{10} - 6\tilde{x}_{11} :\ \tilde{x} \in \{0,1\}^2 \right\}.$$

The simplest SDP relaxation of this 0-1 QP is

$$\begin{array}{lll} \min & \tilde{X}_{1010} + 4\tilde{X}_{1011} + 4\tilde{X}_{1111} - 3\tilde{x}_{10} - 6\tilde{x}_{11} & \\ \text{s.t.} & \tilde{x}_{1s} = \tilde{X}_{1s1s} & (s = 0,1) \\ & \tilde{X}^+ = \begin{pmatrix} 1 & \tilde{x}_{10} & \tilde{x}_{11} \\ \tilde{x}_{10} & \tilde{X}_{1010} & \tilde{X}_{1011} \\ \tilde{x}_{11} & \tilde{X}_{1110} & \tilde{X}_{1111} \end{pmatrix} \succeq 0. & \end{array}$$

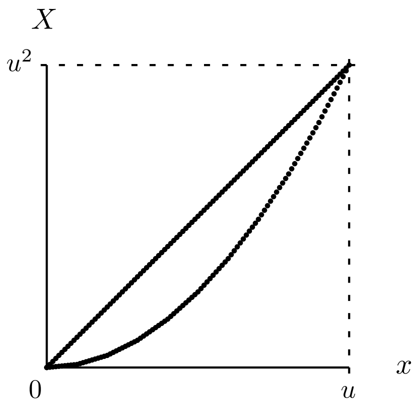

Figure 1: The convex set $F(u)$.

One can check with an SDP solver that all optimal solutions to this SDP have cost -2. So, the transformed SDP yields an improved (in fact optimal) lower bound for this instance. □

The reason for the improvement in the bound appears to be the constraint $\text{diag}(\tilde{X}) = \tilde{x}$, which has no analog in the general-integer case. The following lemma and theorem explore this phenomenon in more detail.

Lemma 1 *Let u_1 be a positive integer. Consider a trivial IQP of the form*

$$\min \left\{ Q_{11} x_1^2 + c_1 x_1 : \; x_1 \in \mathbb{Z} \cap [0, u_1] \right\}, \tag{9}$$

along with its SDP relaxation

$$\min \left\{ Q_{11} X_{11} + c_1 x_1 : \; x_1 \in [0, u_1], \; X_{11} \le u_1 x_1, \; X^+ \succeq 0 \right\}. \tag{10}$$

Let $F(u)$ denote the feasible region of this SDP. We have:

$$F(u) = \textit{conv} \left\{ (x_1, X_{11}) \in \mathbb{R}^2 : \; 0 \le x_1 \le u_1, \; X_{11} = x_1^2 \right\}.$$

Proof. This follows from the fact that X^+ is psd if and only if its determinant is non-negative, i.e., if and only if $X_{11} \ge x_1^2$ (see Figure 1). □

In other words, the SDP relaxation (10) does not exploit the integrality of x_1 in any way.

Theorem 2 *Consider again the trivial IQP (9). Suppose we apply the pro-*

cedure in Theorem 1 to the SDP relaxation (10), yielding the SDP relaxation

$$\begin{aligned}
\min \quad & Q_{11} \sum_{s=0}^{r_1} \sum_{t=0}^{r_1} 2^{s+t} \tilde{X}_{1s1t} + c_1 \sum_{s=0}^{r_1} 2^s \tilde{x}_{1s} \\
s.t. \quad & 0 \leq \sum_{s=0}^{r_1} 2^s \tilde{x}_{1s} \leq u_1 \\
& \sum_{s=0}^{r_1} \sum_{t=0}^{r_1} 2^{s+t} \tilde{X}_{1s1t} \leq u_1 \sum_{s=0}^{r_1} 2^s \tilde{x}_{1s} \\
& \mathrm{diag}(\tilde{X}) = \tilde{x} \\
& \tilde{X}^+ = \begin{pmatrix} 1 & \tilde{x}^T \\ \tilde{x} & \tilde{X} \end{pmatrix} \succeq 0.
\end{aligned}$$

Now, suppose that we project the feasible region of this SDP onto a two-dimensional subspace having x_1 and X_{11} as axes, using the linear mappings

$$x_1 = \sum_{s=0}^{r_1} 2^s \tilde{x}_{1s}$$

and

$$X_{11} = \sum_{s=0}^{r_i} \sum_{t=0}^{r_i} 2^{s+t} \tilde{X}_{1s1t}.$$

Then the projection satisfies the linear inequality

$$X_{11} \geq \bar{u} x_1 - \frac{(\bar{u}^2 - 1)}{4}, \tag{11}$$

where $\bar{u} = \sum_{s=0}^{r_1} 2^s = 2^{r_1+1} - 1$. (See Figure 2 for an illustration for the case $u_1 = \bar{u} = 3$.)

Proof. For ease of notation, let us drop the index 1, writing r for r_1, $\tilde{x}_s$ for $\tilde{x}_{1s}$ and $\tilde{X}_{st}$ for $\tilde{X}_{1s1t}$. Let $v \in \mathbb{Z}_+^{r+1}$ be the vector with

- $v_s = 2^s$ for $s = 0, \ldots, r-1$;
- $v_r = 2^r - 1$.

Also let $w = 2^r - 1 = \lfloor \bar{u}/2 \rfloor$. Since $\tilde{X}^+$ is psd, we have:

$$\begin{pmatrix} -w & v^T \end{pmatrix} \begin{pmatrix} 1 & \tilde{x}^T \\ \tilde{x} & \tilde{X} \end{pmatrix} \begin{pmatrix} -w \\ v \end{pmatrix} \geq 0,$$

or, equivalently, $v^T \tilde{X} v \geq (2w) v^T \tilde{x} - w^2$. Expanding this gives:

$$\sum_{s=0}^{r} \sum_{t=0}^{r} v_s v_t \tilde{X}_{st} \geq (\bar{u} - 1) \sum_{s=0}^{r} v_s \tilde{x}_s - w^2. \tag{12}$$

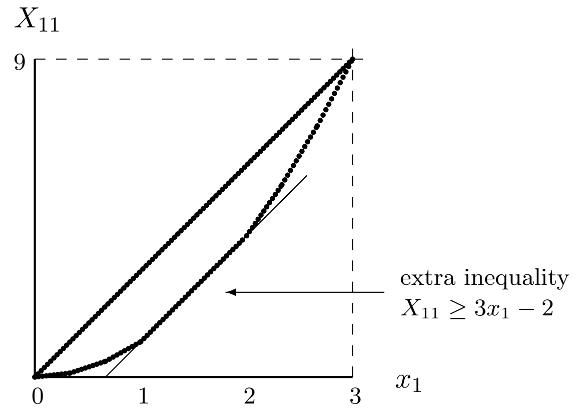

Figure 2: Projection of SDP relaxation with bit representation when $u_1 = 3$.

Next, observe that, for $s = 0, \ldots, r-1$, we have

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{x}_s & \tilde{x}_r \\ \tilde{x}_s & \tilde{X}_{ss} & \tilde{X}_{sr} \\ \tilde{x}_r & \tilde{X}_{sr} & \tilde{X}_{rr} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \geq 0,$$

which, together with the equation $\mathrm{diag}(\tilde{X}) = \tilde{x}$, implies $2\tilde{X}_{sr} \geq \tilde{x}_s + \tilde{x}_r - 1$. Multiplying these last inequalities by 2^s and summing over all s yields:

$$\sum_{s=0}^{r-1} 2^{s+1}\tilde{X}_{sr} \geq \sum_{s=0}^{r-1} 2^s \tilde{x}_s + w x_r - w. \tag{13}$$

Finally, summing the inequalities (12) and (13), along with the trivial inequality $\bar{u}\,\tilde{X}_{st} \geq 0$, we obtain

$$\sum_{s=0}^{r}\sum_{t=0}^{r} 2^{s+t}\,\tilde{X}_{st} \geq \bar{u}\sum_{s=0}^{r} 2^s \tilde{x}_s - w(w+1),$$

which is equivalent to (11). □

We remark that an alternative (but longer) proof of Theorem 2 can be derived by using Theorem 3.2 of Delorme & Poljak [10] along with the so-called 'covariance mapping' (see Deza & Laurent [11], Section 5.2). We omit the details for the sake of brevity.

References

[1] W.P. Adams & H.D. Sherali (1986) A tight linearization and an algorithm for zero-one quadratic programming problems. *Mngt. Sci.*, 32, 1274–1290.

[2] K. Anstreicher (2009) Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. *J. Glob. Optim.*, 43, 471–484.

[3] A. Billionnet, S. Elloumi & A. Lambert (2012) Extending the QCR method to general mixed-integer programs. *Math. Program.*, 131, 381–401.

[4] P. Bonami & F. Margot (2015) Cut generation through binarization. *Math. Program.*, 154, 197–223.

[5] C. Buchheim & A. Wiegele (2013) Semidefinite relaxations for non-convex quadratic mixed-integer programming. *Math. Program.*, 141, 435–452.

[6] S. Burer & A.N. Letchford (2009) On non-convex quadratic programming with box constraints. *SIAM J. Optim.*, 20, 1073–1089.

[7] S. Burer & A.N. Letchford (2012) Non-convex mixed-integer nonlinear programming: a survey. *Surveys in Oper. Res. & Mgmt. Sci.*, 17, 97–106.

[8] S. Burer & A.N. Letchford (2014) Unbounded convex sets for non-convex mixed-integer quadratic programming. *Math. Program.*, 143, 231–256.

[9] S. Burer & A. Saxena (2011) The MILP road to MIQCP. In: J. Lee & S. Leyffer (eds.) *Mixed-Integer Nonlinear Programming*, pp. 373–406. IMA Volumes in Mathematics and its Applications, vol. 154. New York: Springer US.

[10] C. Delorme & S. Poljak (1993) Laplacian eigenvalues and the maximum cut problem. *Math. Program.*, 62, 557–574.

[11] M.M. Deza & M. Laurent (1997) *Geometry of Cuts and Metrics.* Berlin: Springer.

[12] A. Faye & F. Roupin (2007) Partial Lagrangian relaxation for general quadratic programming. *4OR*, 5, 75–88.

[13] T. Fujie & M. Kojima (1997) Semidefinite programming relaxation for nonconvex quadratic programs. *J. Glob. Optim.*, 10, 367–380.

[14] L. Galli, D. Grainger & A.N. Letchford (2016) Using bit representation to improve LP relaxations of mixed-integer quadratic programs. *Working paper*, Department of Management Science, Lancaster University.

[15] L. Galli, K. Kaparis & A.N. Letchford (2011) Gap inequalities for non-convex mixed-integer quadratic programs. *Oper. Res. Lett.*, 39, 297–300.

[16] A. Gupte, S. Ahmed, M.S. Cheon & S. Dey (2013) Solving mixed integer bilinear problems using MILP formulations. *SIAM J. Optim.*, 23, 721–744.

[17] C. Helmberg & F. Rendl (1998) Solving quadratic (0,1)-problems by semidefinite programs and cutting planes. *Math. Program.*, 82, 291–315.

[18] C. Helmberg, S. Poljak, F. Rendl & H. Wolkowicz (1995) Combining semidefinite and polyhedral relaxations for integer programs. In E. Balas & J. Clausen (eds) *Integer Programming and Combinatorial Optimization IV*, pp. 124–134. Lecture Notes in Computer Science, vol. 920. Berlin: Springer.

[19] F. Körner (1988) A tight bound for the Boolean quadratic optimization problem and its use in a branch and bound algorithm. *Optimization*, 19, 711–721.

[20] M. Laurent (2003) A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0-1 programming. *Math. Oper. Res.*, 28, 470–496.

[21] M. Laurent & F. Rendl (2005) Semidefinite programming and integer programming. In K. Aardal, G.L. Nemhauser & R. Weismantel (eds) *Discrete Optimization*, pp. 393–514. Handbooks in OR and MS, vol. 12. Amsterdam: Elsevier.

[22] C. Lemaréchal & F. Oustry (2001) SDP relaxations in combinatorial optimization from a Lagrangian viewpoint. In N. Hadjisawas & P.M. Pardalos (eds.), *Advances in Convex Analysis and Global Optimization.* Dortrecht: Kluwer.

[23] L. Lovász (1979) On the Shannon capacity of a graph. *IEEE Trans. Inform. Th.*, IT-25, 1–7.

[24] L. Lovász & A.J. Schrijver (1991) Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optim.*, 1, 166–190.

[25] F.M. Muldoon, W.P. Adams & H.D. Sherali (2013) Ideal representations of lexicographic orderings and base-2 expansions of integer variables. *Oper. Res. Lett.*, 41, 32–39.

[26] J.H. Owen & S. Mehrotra (2002) On the value of binary expansions for general mixed-integer linear programs. *Oper. Res.*, 50, 810–819.

[27] S. Poljak, F. Rendl & H. Wolkowicz (1995) A recipe for semidefinite relaxation for (0,1)-quadratic programming. *J. Glob. Optim.*, 7, 51–73.

[28] M. Ramana (1993) *An Algorithmic Analysis of Multiquadratic and Semidefinite Programming Problems.* PhD Thesis, Johns Hopkins University, Baltimore, MD.

[29] J.-S. Roy (2007) "Binarize and project" to generate cuts for general mixed-integer programs. *Alg. Oper. Res.*, 2, 37–51.

[30] H.D. Sherali & C.H. Tuncbilek (1992) A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique. *J. Glob. Optim.*, 2, 101–112.

[31] N.Z. Shor (1987) Quadratic optimization problems. *Sov. J. Comput. Syst. Sci.*, 25, 1–11.

[32] N.Z. Shor (1990) Dual quadratic estimates in polynomial and Boolean programming. *Ann. Oper. Res.*, 25, 163–168.

[33] L.J. Watters (1967) Reduction of integer polynomial programming problems to zero-one linear programming problems. *Oper. Res.*, 15, 1171–1174.