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Convergence of the fixed point algorithm for quasi-equilibria

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Abstract. An algorithm for solving quasi-equilibrium problems (QEPs) is proposed relying on the sequential inexact resolution of equilibrium problems. First, we reformulate QEP as the fixed point problem of a set-valued map and analyse its Lipschitz continuity under strong monotonicity assumptions. Then, a few classes of QEPs satisfying these assumptions are identified. Finally, we devise an algorithm that computes an inexact solution of an equilibrium problem at each iteration and we prove its global convergence.

Keywords.

1 Introduction

In this paper we focus on the following (abstract) *quasi-equilibrium problem*

$$\text{find } \ x^* \in C(x^*) \quad \text{s.t.} \quad f(x^*, y) \geq 0 \quad \text{for all } y \in C(x^*), \tag{QEP}$$

where the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the equilibrium condition $f(x, x) = 0$ for any $x \in \mathbb{R}^n$ and the constraints are given by a set-valued map $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that describes how the feasible region changes together with the considered point.

Throughout the paper we assume that $C(x)$ is closed and convex for any $x \in \mathbb{R}^n$, f is continuous and there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ-convex on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$, that is the function $f(x, \cdot) - \tau \| \cdot \|^2/2$ is convex on $\mathbb{R}^n$.

Fixed point algorithm

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0. Choose $x^0 \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and set $k = 0$.
1. Find $x^{k+1} = \arg\min \left\{ f(x^k, y) + \alpha \|y - x^k\|^2/2 \ : \ y \in C(x^k) \right\}$
2. If $x^{k+1} = x^k$ then stop.
3. Set $k = k + 1$ and go to Step 1.

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2 Monotonicity and Lipschitz assumptions

Throughout the paper the following assumptions will be used:

(M) f is μ-monotone on $\mathbb{R}^n$ for some $\mu \in \mathbb{R}$, i.e. $f(x,y) + f(y,x) \leq -\mu\|x-y\|^2$ holds for any $x, y \in \mathbb{R}^n$;

(PM) f is ρ-pseudomonotone on $\mathbb{R}^n$ for some $\rho \in \mathbb{R}$, i.e. the implication

$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq -\rho\|x-y\|^2$$

holds for any $x, y \in \mathbb{R}^n$;

(MG) $f(x,\cdot)$ is continuously differentiable on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$ and the map $\nabla_2 f(\cdot, w)$ is ν-monotone for any $w \in \mathbb{R}^n$ for some $\nu \in \mathbb{R}$, i.e.

$$\langle \nabla_2 f(u,w) - \nabla_2 f(v,w), u-v \rangle \geq \nu\|u-v\|^2$$

holds for any $u, v, w \in \mathbb{R}^n$;

(T) there exist $T_1 > 0$ and $T_2 > 0$ such that

$$f(x,z) \leq f(x,y) + f(y,z) + T_1\|x-y\|^2 + T_2\|y-z\|^2$$

holds for any $x, y, z \in \mathbb{R}^n$;

(LG) $f(x,\cdot)$ is continuously differentiable on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$ and the map $\nabla_2 f(\cdot, w)$ is L-Lipschitz continuous on $\mathbb{R}^n$ for any $w \in \mathbb{R}^n$, i.e. $\|\nabla_2 f(u,w) - \nabla_2 f(v,w)\| \leq L\,\|u-v\|$ hold for any $u, v, w \in \mathbb{R}^n$;

Some relationships between the above assumptions hold.

Proposition 2.1.

a) If (LG) holds, then (MG) is satisfied with $\nu = -L$;

b) If (LG) holds, then (T) is satisfied with $T_1 = T_2 = L/2$;

c) If (MG) holds, then (M) is satisfied with $\mu = \nu$;

d) If (M) and (T) hold, then $\mu \leq T_1 + T_2$;

e) If (MG) and (LG) hold, then $-L \leq \nu \leq L$;

f) If $f(x,y) = \langle F(x) + Qy, y-x \rangle$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$, then

- *$\tau = \lambda_{min}(Q + Q^T)$,*
- *(M) holds iff the map $x \mapsto F(x) - Qx$ is μ-monotone,*
- *(MG) holds iff (M) holds,*

- *(T) holds with $T_1 = T_2 = (L_F + \|Q\|)/2$ and (LG) holds with $L = (L_F + \|Q\|)$ if F is L_F-Lipschitz continuous,*
- *If, in addition, $F(x) = Px + r$ for some $P \in \mathbb{R}^{n\times n}$ and $r \in \mathbb{R}^n$, then*
  (M) and (MG) hold with $\mu = \nu = \lambda_{min}([P - Q + (P - Q)^T]/2)$,
  (T) holds with $T_1 = T_2 = \|P - Q^T\|/2$,
  (LG) holds with $L = \|P - Q^T\|$.

Proof. a) For any $x, y, z \in \mathbb{R}^n$ we have

$$\langle \nabla_2 f(x,z) - \nabla_2 f(y,z), x - y\rangle \geq -\|\nabla_2 f(x,z) - \nabla_2 f(y,z)\|\|x - y\| \geq -L\|x - y\|^2,$$

b) Let $x, y, z \in \mathbb{R}^n$ and consider the function

$$g(t) = f(y, z + t(y - z)) - f(x, z + t(y - z))$$

for $t \in [0,1]$. Then, assumption (LG) implies

$$\begin{aligned} g'(t) &= \langle \nabla_2 f(y, z + t(y - z)) - \nabla_2 f(x, z + t(y - z)), y - z\rangle \\ &\leq \|\nabla_2 f(y, z + t(y - z)) - \nabla_2 f(x, z + t(y - z))\|\|y - z\| \\ &\leq L\|x - y\|\|y - z\| \\ &\leq L(\|x - y\|^2 + \|y - z\|^2)/2 \end{aligned}$$

for any $t \in [0,1]$. Therefore,

$$\begin{aligned} f(x,z) - f(x,y) - f(y,z) &= g(1) - g(0) \\ &= \int_0^1 g'(t)dt \\ &\leq L(\|x - y\|^2 + \|y - z\|^2)/2, \end{aligned}$$

hence (T) holds with $T_1 = T_2 = L/2$.

c) Let $x, y \in \mathbb{R}^n$ and consider the function

$$g(t) = f(x, x + t(y - x)) - f(y, x + t(y - x))$$

for $t \in [0,1]$. Then, assumption (MG) implies

$$g'(t) = \langle \nabla_2 f(x, x + t(y - x)) - \nabla_2 f(y, x + t(y - x)), y - x\rangle \leq -\nu\|y - x\|^2$$

for any $t \in [0,1]$. Therefore,

$$\begin{aligned} f(x,y) + f(y,x) &= f(x,y) - f(y,y) - f(x,x) + f(y,x) \\ &= g(1) - g(0) \\ &= \int_0^1 g'(t)dt \\ &\leq -\nu\|y - x\|^2, \end{aligned}$$

hence (M) holds with $\mu = \nu$.

d) For any $x, y \in \mathbb{R}^n$, with $x \neq y$,

$$0 = f(x,x) \leq f(x,y) + f(y,x) + (T_1 + T_2)\|x-y\|^2 \leq (T_1 + T_2 - \mu)\|x-y\|^2,$$

thus dividing by $\|x-y\|^2$ we get $T_1 + T_2 \geq \mu$.

e) It follows from b) that $\nu \geq -L$. Moreover, we have

$$\nu\|x-y\|^2 \leq \langle \nabla_2 f(x,z) - \nabla_2 f(y,z), x-y \rangle \leq \|\nabla_2 f(x,z) - \nabla_2 f(y,z)\|\|x-y\| \leq L\|x-y\|^2.$$

f) Since

$$f(x,y) = y^T Q y + y^T[F(x) - Q^T x] - x^T F(x) = \frac{1}{2} y^T(Q+Q^T)y + y^T[F(x) - Q^T x] - x^T F(x),$$

the function $f(x,\cdot)$ is τ-convex with $\tau = \lambda_{min}(Q+Q^T)$.

Since

$$f(x,y) + f(y,x) = \langle F(x) - Qx - [F(y) - Qy], y-x \rangle, \tag{1}$$

the bifunction f is μ-monotone if and only if the map $x \mapsto F(x) - Qx$ is μ-monotone.

Since $\nabla_2 f(x,y) = F(x) - Q^T x + (Q+Q^T)y$, condition (MG) holds if and only if the map $x \mapsto F(x) - Q^T x$ is ν-monotone or equivalently $x \mapsto F(x) - Qx$ is ν-monotone. Hence, (M) and (MG) are equivalent with the same monotonicity modulus.

If F is L_F-Lipschitz continuous, then

$$\begin{aligned} f(x,z) - f(x,y) - f(y,z) &= \langle F(x) - Q^T x - [F(y) - Q^T y], z-y \rangle \\ &\leq (L_F + \|Q\|)\, \|x-y\|\|y-z\| \\ &\leq (L_F + \|Q\|)\, (\|x-y\|^2 + \|y-z\|^2)/2, \end{aligned}$$

and

$$\begin{aligned} \|\nabla_2 f(x,y) - \nabla_2 f(z,y)\| &= \|F(x) - F(z) + Q^T(z-x)\| \\ &\leq (L_F + \|Q\|)\|x-z\| \end{aligned}$$

hence (T) holds with $T_1 = T_2 = (L_F + \|Q\|)/2$ and (LG) holds with $L = (L_F + \|Q\|)$.

Finally, if $F(x) = Px + r$, then (1) implies (M) and (MG) hold with the modulus given by the minimum eigenvalue of the symmetric part of $P - Q$. Moreover,

$$\begin{aligned} f(x,z) - f(x,y) - f(y,z) &= \langle (P - Q^T)(x-y), z-y \rangle \\ &\leq \|P - Q^T\|\, \|x-y\|\|y-z\| \\ &\leq \|P - Q^T\|\, (\|x-y\|^2 + \|y-z\|^2)/2, \end{aligned}$$

and

$$\begin{aligned} \|\nabla_2 f(x,y) - \nabla_2 f(z,y)\| &= \|(P - Q^T)(x-z)\| \\ &\leq \|P - Q^T\|\|x-z\| \end{aligned}$$

hence (T) holds with $T_1 = T_2 = \|P - Q^T\|/2$ and (LG) holds with $L = \|P - Q^T\|$.

□

3 Convergence under (LB)

3.1 Lipschitz behaviour of minima

Assumption (LB):
given $\alpha \in \mathbb{R}$, there exists $\Lambda(\alpha) \geq 0$ such that $\|y_\alpha(x,z) - y_\alpha(x,z')\| \leq \Lambda(\alpha)\,\|z - z'\|$ holds for any $x, z, z' \in \mathbb{R}^n$, where

$$y_\alpha(x,z) := \arg\min\left\{ f(x,y) + \frac{\alpha}{2}\|y - x\|^2 \ : \ y \in C(z) \right\}.$$

When (QEP) is a (QVI), i.e., $f(x,y) = \langle F(x), y - x\rangle$, (LB) holds if there exists $\Lambda(\alpha) \geq 0$ such that

$$\|P_{C(z)}(x) - P_{C(z')}(x)\| \leq \Lambda(\alpha)\|z - z'\|$$

hold for any $x, z, z' \in \mathbb{R}^n$, where P_X denotes the Euclidean projection on the set X.

Assumption (LB) is satisfied when the feasible set is a moving/expanding set...

Proposition 3.1. *Let*

$$C(x) = s(x)K + t(x) \tag{2}$$

where $K \subset \mathbb{R}^n$ is closed and convex, $t : \mathbb{R}^n \to \mathbb{R}^n$ is L_t-Lipschitz continuous, $s : \mathbb{R}^n \to \mathbb{R}$ is L_s-Lipschitz continuous and $s(x) > 0$ holds for any $x \in \mathbb{R}^n$. Suppose that $f(x,\cdot)$ is continuously differentiable and the map $\nabla_2 f(x,\cdot)$ is L_2-Lipschitz continuous for any $x \in \mathbb{R}^n$.

If s is constant, then (LB) holds with

$$\Lambda(\alpha) = \frac{L_t(|\alpha| + L_2)}{\alpha + \tau}.$$

for any $\alpha > -\tau$.

If K is bounded then (LB) holds with

$$\Lambda(\alpha) = \frac{(RL_s + L_t)(|\alpha| + L_2)}{\alpha + \tau},$$

for any $\alpha > -\tau$, where $R \geq \max_{x \in K}\|x\|$.

Proof. Given $x \in \mathbb{R}^n$, consider the function $p(y) = f(x,y) + \alpha\|y - x\|^2/2$. By definition, $y_\alpha(x,z) = \arg\min\{p(y) : \ y \in C(z)\}$ and $y_\alpha(x,z') = \arg\min\{p(y) : \ y \in C(z')\}$. Then the first-order optimality conditions for $y_\alpha(x,z)$ and $y_\alpha(x,z')$ read

$$\langle \nabla p(y_\alpha(x,z)), y - y_\alpha(x,z)\rangle \geq 0, \qquad \forall\, y \in C(z),$$
$$\langle \nabla p(y_\alpha(x,z')), y - y_\alpha(x,z')\rangle \geq 0, \qquad \forall\, y \in C(z').$$

Since $s(z)[y_\alpha(x,z') - t(z')]/s(z') + t(z) \in C(z)$ and $s(z')[y_\alpha(x,z) - t(z)]/s(z) + t(z') \in C(z')$, summing the above inequalities with these choices of y, we obtain

$$\langle \nabla p(y_\alpha(x,z)) - \nabla p(y_\alpha(x,z')), s(z)[y_\alpha(x,z') - t(z')] + s(z')[t(z) - y_\alpha(x,z)]\rangle \geq 0. \tag{3}$$

Since p is $(\alpha+\tau)$-convex, the map ∇p is $(\alpha+\tau)$-monotone and Lipschitz continuous with constant $(|\alpha|+L_2)$ by assumption (LG2). Hence, we get

$$\begin{aligned}
&(\alpha+\tau)s(z')\|y_\alpha(x,z)-y_\alpha(x,z')\|^2 \\
&\quad\le \langle \nabla p(y_\alpha(x,z))-\nabla p(y_\alpha(x,z')), s(z')\,[y_\alpha(x,z)-y_\alpha(x,z')]\rangle \\
&\quad\le \langle \nabla p(y_\alpha(x,z))-\nabla p(y_\alpha(x,z')), [s(z)-s(z')][y_\alpha(x,z')-t(z')]+s(z')[t(z)-t(z')]\rangle \\
&\quad\le \|\nabla p(y_\alpha(x,z))-\nabla p(y_\alpha(x,z'))\|\ \ [\|s(z)-s(z')\|\|y_\alpha(x,z')-t(z')\|+s(z')\|t(z)-t(z')\|] \\
&\quad\le (L_2+|\alpha|)\,[L_s\|y_\alpha(x,z')-t(z')\|+s(z')L_t]\,\|y_\alpha(x,z)-y_\alpha(x,z')\|\,\|z-z'\|,
\end{aligned}$$

where the second inequality follows from (3), the third from the Cauchy-Schwarz inequality and the last from the assumptions. Therefore, we have

$$(\alpha+\tau)s(z')\|y_\alpha(x,z)-y_\alpha(x,z')\|\le (L_2+|\alpha|)[L_s\|y_\alpha(x,z')-t(z')\|+s(z')L_t]\|z-z'\|.$$

If $L_s=0$, then the thesis follows dividing both members of the above inequality by $s(z')$. Otherwise, if K is bounded, we get $\|y_\alpha(x,z')-t(z')\|\le Rs(z')$ since $y_\alpha(x,z')-t(z')\in s(z')K$. Dividing by $s(z')$, we obtain the thesis. □

In the case of a QVI, under the assumptions of Proposition 3.1, condition (LB) holds with the constant $\Lambda(\alpha)=RL_s+L_t$ which does not depend on parameter α.

Remark 3.1. *If $K=B(0,R)$ and $s(x)=\gamma\|x\|$, then a point $x\neq 0$ is a fixed point of the set-valued map C if and only if $\gamma R\ge 1$. Anyway, if QEP is actually a QVI the $\Lambda(\alpha)=RL_s=R\gamma$ holds. The convergence of the fixed-point algorithm requires $\Lambda(\alpha)<1$, that is 0 is the unique fixed point of C and the problem is not really meaningful.*

(LB) is satisfied also in the case of linear constraints with variable right-hand side.

Lemma 3.1. *Let $X\subseteq\mathbb{R}^m$ be a closed convex set and $g:\mathbb{R}^m\times\mathbb{R}^p\to\mathbb{R}$ a function such that $g(\cdot,v)$ is differentiable and τ-convex with $\tau>0$ for any $v\in\mathbb{R}^p$ and $\nabla_1 g(u,\cdot)$ is Lipschitz continuous on $\mathbb{R}^p$ with constant L for any $u\in\mathbb{R}^m$. Then, the function*

$$x(v)=\arg\min\{g(u,v):\ u\in X\}$$

is Lipschitz continuous on $\mathbb{R}^p$ with constant L/τ.

Proof. It follows from the assumptions that the function $x(v)$ is well defined. Let $v,v'\in\mathbb{R}^p$ be fixed. Then the first-order optimality conditions imply that

$$\langle \nabla_1 g(x(v),v), u-x(v)\rangle\ge 0,\qquad \forall\ u\in X, \tag{4}$$

$$\langle \nabla_1 g(x(v'),v'), u-x(v')\rangle\ge 0,\qquad \forall\ u\in X. \tag{5}$$

If we set $u = x(v')$ in (4), $u = x(v)$ in (5) and sum the two inequalities, we get

$$
\begin{aligned}
0 &\le \langle \nabla_1 g(x(v'), v') - \nabla_1 g(x(v), v), x(v) - x(v') \rangle \\
&= \langle \nabla_1 g(x(v'), v') - \nabla_1 g(x(v'), v), x(v) - x(v') \rangle \\
&\quad + \langle \nabla_1 g(x(v'), v) - \nabla_1 g(x(v), v), x(v) - x(v') \rangle \\
&\le \langle \nabla_1 g(x(v'), v') - \nabla_1 g(x(v'), v), x(v) - x(v') \rangle - \tau \|x(v) - x(v')\|^2 \\
&\le \|\nabla_1 g(x(v'), v') - \nabla_1 g(x(v'), v)\| \|x(v) - x(v')\| - \tau \|x(v) - x(v')\|^2 \\
&\le L\|v - v'\| \|x(v) - x(v')\| - \tau \|x(v) - x(v')\|^2,
\end{aligned}
$$

where the second inequality follows from the τ-monotonicity of $\nabla_1 g(\cdot, v)$ for any $v \in \mathbb{R}^p$ and the last one from the Lipschitz assumption on $\nabla_1 g(u, \cdot)$. Therefore, we obtain

$$\|x(v) - x(v')\| \le \frac{L}{\tau} \|v - v'\|.$$

□

Proposition 3.2. *Suppose that $f(x, y) = \langle F(x) + Qy, y - x \rangle$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite, the set*

$$C(x) = \{y \in \mathbb{R}^n : \; Ay \le b(x)\}, \tag{6}$$

where $b : \mathbb{R}^n \to \mathbb{R}^m$ is L_b-Lipschitz continuous and $A \in \mathbb{R}^{m \times n}$ with $m \le n$ and $rank(A) = m$. If $\alpha > -\tau$, where $\tau = 2\lambda_{min}(Q)$, then assumption (LB) is satisfied with

$$\Lambda(\alpha) = \frac{\|A\| L_b}{(\alpha + \tau) \lambda_{min}(A(2Q + \alpha I)^{-1} A^T)}.$$

Proof. Let $x \in \mathbb{R}^n$ be fixed and consider the function $p(y) = f(x, y) + \alpha \|y - x\|^2/2$. By definition, $y_\alpha(x, z) = \arg\min\{p(y) : \; Ay \le b(z)\}$ and $y_\alpha(x, z') = \arg\min\{p(y) : \; Ay \le b(z')\}$. We denote $y_z = y_\alpha(x, z)$ and $y_{z'} = y_\alpha(x, z')$. Then the KKT conditions for y_z and $y_{z'}$ imply that there exist multipliers vectors $\lambda_z, \lambda_{z'} \in \mathbb{R}^m_+$ such that

$$\nabla p(y_z) + A^T \lambda_z = 0, \qquad \nabla p(y_{z'}) + A^T \lambda_{z'} = 0.$$

Since p is $(\alpha + \tau)$-convex, the map ∇p is $(\alpha + \tau)$-monotone, we have

$$
\begin{aligned}
(\alpha + \tau) \|y_z - y_{z'}\|^2 &\le (\nabla p(y_z) - \nabla p(y_{z'}))^T (y_z - y_{z'}) \\
&= (\lambda_z - \lambda_{z'})^T A (y_z - y_{z'}) \\
&\le \|\lambda_z - \lambda_{z'}\| \|A\| \|y_z - y_{z'}\|,
\end{aligned}
$$

hence

$$\|y_z - y_{z'}\| \le \frac{\|A\|}{\alpha + \tau} \|\lambda_z - \lambda_{z'}\|. \tag{7}$$

Now, let us consider the Lagrangian dual problem of

$$\min\{p(y) : \; Ay \le b(z)\}. \tag{8}$$

The Lagrangian function is

$$\begin{aligned} L(y,\lambda) &= p(y) + \lambda^T(Ay - b(z)) \\ &= \frac{1}{2} y^T(2Q+\alpha I)y + y^T(F(x) - Qx - \alpha x + A^T\lambda) + \frac{\alpha}{2}\|x\|^2 - x^T F(x) - \lambda^T b(z). \end{aligned}$$

Since the matrix $2Q + \alpha I$ is positive definite, we get

$$\arg\min_{y \in \mathbb{R}^n} L(y,\lambda) = -(2Q+\alpha I)^{-1}(F(x) - Qx - \alpha x + A^T\lambda).$$

Thus the Lagrangian dual problem of (8) is

$$\begin{aligned} \max_{\lambda \geq 0} \quad & \left\{ -\frac{1}{2}\lambda^T A(2Q+\alpha I)^{-1}A^T\lambda - \lambda^T[b(z) + A(F(x) - Qx - \alpha x)] \right. \\ & \left. +\frac{\alpha}{2}\|x\|^2 - x^T F(x) \right\}. \end{aligned} \tag{9}$$

Since λ_z is an optimal solution to (9), we have

$$\lambda_z = \arg\min\{g(\lambda, z) : \ \lambda \geq 0\},$$

where

$$g(\lambda, z) = \frac{1}{2}\lambda^T A(2Q+\alpha I)^{-1}A^T\lambda + \lambda^T[b(z) + A(F(x) - Qx - \alpha x)].$$

Similarly, it can be proved that $\lambda_{z'} = \arg\min\{g(\lambda, z') : \ \lambda \geq 0\}$. Since $2Q + \alpha I$ is positive definite and $rank(A) = m$, the matrix $A(2Q + \alpha I)^{-1}A^T$ is positive definite. If we denote $\sigma = \lambda_{min}(A(2Q + \alpha I)^{-1}A^T)$, we obtain $g(\cdot, z)$ is σ-convex for any $z \in \mathbb{R}^n$ and $\nabla_1 g(\lambda, \cdot)$ is Lipschitz continuous with constant L_b for any $\lambda \geq 0$. Therefore, Lemma 3.1 guarantees that

$$\|\lambda_z - \lambda_{z'}\| \leq \frac{L_b}{\sigma}\|z - z'\|.$$

Finally, the above inequality together with (7) imply

$$\|y_z - y_{z'}\| \leq \frac{\|A\| L_b}{(\alpha+\tau)\sigma}\|z - z'\|.$$

□

In the case of a QVI, under the assumptions of Proposition 3.2, condition (LB) holds with the constant $\Lambda(\alpha) = \|A\| L_b / \lambda_{min}(AA^T)$ which does not depend on parameter α.

3.2 Convergence of the algorithm

Theorem 3.1. *Suppose that (MG), (LG) and (LB) hold for some $\alpha > -\tau$. Then:*

a) the map $S_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$S_\alpha(x) = \arg\min\left\{ f(x,y) + \frac{\alpha}{2}\|y - x\|^2 \ : \ y \in C(x) \right\}.$$

is Lipschitz continuous on $\mathbb{R}^n$ *with constant* $\Lambda(\alpha) + r_1(\alpha)$, *where*

$$r_1(\alpha) = \begin{cases} \dfrac{L-\alpha}{\alpha+\tau} & \text{if } \alpha \in (-\tau, 0], \\[2ex] \dfrac{\sqrt{\alpha^2 - 2\alpha\nu + L^2}}{\alpha+\tau} & \text{if } \alpha > 0. \end{cases} \tag{10}$$

b) If $\Lambda(\alpha)+r_1(\alpha) < 1$, *then there exists a unique solution* $\bar{x}$ *to (QEP) and the sequence* $\{x^k\}$ *generated by the fixed-point algorithm converges to* $\bar{x}$ *with the linear rate of convergence given by*

$$\|x^{k+1} - \bar{x}\| \leq [\Lambda(\alpha) + r_1(\alpha)]\|x^k - \bar{x}\| \qquad \forall\ k \in \mathbb{N}. \tag{11}$$

Proof. a) Since the function $y \mapsto f(x,y) + \alpha\|y-x\|^2/2$ is $(\alpha+\tau)$-convex and $\alpha > -\tau$, the point $S_\alpha(x)$ is unique for any $x \in \mathbb{R}^n$.

Let $u, v \in \mathbb{R}^n$ be given. Assumption (LB) guarantees that

$$\begin{aligned} \|S_\alpha(u) - S_\alpha(v)\| &= \|y_\alpha(u,u) - S_\alpha(v)\| \\ &\leq \|y_\alpha(u,u) - y_\alpha(u,v)\| + \|y_\alpha(u,v) - S_\alpha(v)\| \\ &\leq \Lambda(\alpha)\|u-v\| + \|y_\alpha(u,v) - S_\alpha(v)\|. \end{aligned} \tag{12}$$

We denote $y = y_\alpha(u,v)$ and $s = S_\alpha(v)$. Applying the first-order optimality conditions to y and s, we have

$$\langle \nabla_2 f(u,y) + \alpha(y-u), z-y\rangle \geq 0, \qquad \forall\ z \in C(v), \tag{13}$$

$$\langle \nabla_2 f(v,s) + \alpha(s-v), z-s\rangle \geq 0, \qquad \forall\ z \in C(v). \tag{14}$$

Setting $z = s$ in (13) and $z = y$ in (14) and summing the two inequalities, we get

$$\langle \nabla_2 f(u,y) - \nabla_2 f(v,s) + \alpha(y-s+v-u), s-y\rangle \geq 0,$$

hence

$$\alpha\|y-s\|^2 \leq \langle \nabla_2 f(u,y) - \nabla_2 f(v,s) + \alpha(v-u), s-y\rangle. \tag{15}$$

Moreover, the τ-convexity of $f(u,\cdot)$ implies the map $\nabla_2 f(u,\cdot)$ is τ-monotone for any u, thus

$$\tau\|y-s\|^2 \leq \langle \nabla_2 f(u,s) - \nabla_2 f(u,y), s-y\rangle. \tag{16}$$

Summing (15) and (16), we get

$$\begin{aligned} (\alpha+\tau)\|y-s\|^2 &\leq \langle \nabla_2 f(u,s) - \nabla_2 f(v,s) + \alpha(v-u), s-y\rangle \\ &\leq \|\nabla_2 f(u,s) - \nabla_2 f(v,s) + \alpha(v-u)\|\|y-s\| \end{aligned}$$

that is

$$(\alpha+\tau)\|y-s\| \leq \|\nabla_2 f(u,s) - \nabla_2 f(v,s) + \alpha(v-u)\|. \tag{17}$$

Therefore, we have

$$\begin{aligned}(\alpha+\tau)^2\|y-s\|^2 &\le \|\nabla_2 f(u,s)-\nabla_2 f(v,s)+\alpha(v-u)\|^2\\ &= \|\nabla_2 f(u,s)-\nabla_2 f(v,s)\|^2+\alpha^2\|v-u\|^2\\ &\quad + 2\alpha\langle\nabla_2 f(u,s)-\nabla_2 f(v,s), v-u\rangle\\ &\le (L^2+\alpha^2)\|u-v\|^2+2\alpha\langle\nabla_2 f(u,s)-\nabla_2 f(v,s), v-u\rangle.\end{aligned}$$

If $\alpha > 0$, then assumptions (MG) implies

$$2\alpha\langle\nabla_2 f(u,s)-\nabla_2 f(v,s), v-u\rangle \le -2\alpha\nu\|u-v\|^2,$$

while if $\alpha \in (-\tau, 0]$ the Cauchy-Schwarz inequality and assumption (LG) imply

$$\begin{aligned}2\alpha\langle\nabla_2 f(u,s)-\nabla_2 f(v,s), v-u\rangle &\le -2\alpha\|\nabla_2 f(u,s)-\nabla_2 f(v,s)\|\|u-v\|\\ &\le -2\alpha L\|u-v\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(\alpha+\tau)^2\|y-s\|^2 \le (\alpha^2+L^2-2\alpha\nu)\|u-v\|^2 && \text{if } \alpha>0,\\ &(\alpha+\tau)^2\|y-s\|^2 \le (L-\alpha)^2\|u-v\|^2 && \text{if } \alpha\in(-\tau,0].\end{aligned}$$

Combining the above inequalities with (12), we get the thesis.

b) If $\Lambda(\alpha)+r_1(\alpha)<1$, then the map S_α is a contraction, hence it has a unique fixed-point $\bar{x}$ which is the unique solution to (QEP) and the linear convergence of the algorithm follows.

□

The above theorem provides an existence and uniqueness result for (QEP) as well.

Remark 3.2. *Spiegare perche' mettiamo sia (MG) che (LG) nonostante la seconda implichi la prima (caso $\nu \neq -L$)... Abbiamo qualcosa di adatto tra gli esempi? Forse Example 4.3?*

Lemma 3.2. *Let $X \subseteq \mathbb{R}^n$ be closed convex set and suppose $h : \mathbb{R}^n \to \mathbb{R}$ is τ-convex with $\tau \ge 0$. Given $\xi \in \mathbb{R}^n$ and $\alpha > -\tau$, then the unique minimizer*

$$x^+ = \arg\min\{h(y)+\alpha\|y-\xi\|^2/2 \;:\; y\in X\}$$

and any $x \in X$ satisfy

$$h(x)+\alpha\|x-\xi\|^2/2 \ge h(x^+)+\alpha\|x^+-\xi\|^2/2+(\alpha+\tau)\|x-x^+\|^2/2. \tag{18}$$

Proof. The objective function $h_\alpha(y)=h(y)+\alpha\|y-\xi\|^2/2$ is $(\alpha+\tau)$-convex with $\alpha+\tau>0$ so that it admits a unique minimizer x^+ over X. Moreover, the convexity of $h_\alpha-(\alpha+\tau)\|\cdot\|^2/2$ guarantees that the inequality

$$h_\alpha(x)-(\alpha+\tau)\|x\|^2/2 \ge h_\alpha(x^+)-(\alpha+\tau)\|x^+\|^2/2+\langle x^*, x-x^+\rangle$$

and equivalently

$$\begin{aligned} h_\alpha(x) &\ge h_\alpha(x^+) + \langle x^*, x - x^+\rangle + (\alpha+\tau)(\|x\|^2 + \|x^+\|^2 - 2\langle x^+, x\rangle)/2 \\ &= h_\alpha(x^+) + \langle x^*, x - x^+\rangle + (\alpha+\tau)\|x - x^+\|^2/2 \end{aligned}$$

hold for any $x \in X$ and any $x^* \in \partial h_\alpha(x^+)$. The optimality of x^+ guarantees that the inequality $\langle x^*, x - x^+\rangle \ge 0$ holds for some $x^* \in \partial h_\alpha(x^+)$ so that the thesis readily follows. □

Theorem 3.2. *Suppose that $\bar{x}$ is a solution of (QEP) and assumptions (M) and (T) are satisfied. If (LB) holds for some $\alpha \ge 2T_2$ and $\Lambda(\alpha) + r_2(\alpha) < 1$, where*

$$r_2(\alpha) = \sqrt{\frac{\alpha + 2(T_1 - \mu)}{\alpha + 2\tau}}, \tag{19}$$

then the sequence $\{x^k\}$ converges to $\bar{x}$ with the linear rate of convergence $\Lambda(\alpha) + r_2(\alpha)$.

Proof. Assumption (LB) guarantees that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} \|x^{k+1} - \bar{x}\| &= \|y_\alpha(x^k, x^k) - \bar{x}\| \\ &\le \|y_\alpha(x^k, x^k) - y_\alpha(x^k, \bar{x})\| + \|y_\alpha(x^k, \bar{x}) - \bar{x}\| \\ &\le \Lambda(\alpha)\|x^k - \bar{x}\| + \|y_\alpha(x^k, \bar{x}) - \bar{x}\|. \end{aligned} \tag{20}$$

Applying Lemma 3.2 with $h = f(x^k, \cdot)$, $\xi = x^k$, $X = C(\bar{x})$, $x^+ = y_\alpha(x^k, \bar{x})$ and $x = \bar{x}$, we get

$$f(x^k, \bar{x}) + \frac{\alpha}{2}\|x^k - \bar{x}\|^2 \ge f(x^k, y_\alpha(x^k, \bar{x})) + \frac{\alpha}{2}\|y_\alpha(x^k, \bar{x}) - x^k\|^2 + \frac{\alpha+\tau}{2}\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2. \tag{21}$$

On the other hand, applying Lemma 3.2 with $h = f(\bar{x}, \cdot)$, $\xi = \bar{x}$, $X = C(\bar{x})$, $x^+ = \bar{x}$ and $x = y_\alpha(x^k, \bar{x})$, and exploiting that $\bar{x}$ solves (QEP), we get

$$f(\bar{x}, y_\alpha(x^k, \bar{x})) + \frac{\alpha}{2}\|\bar{x} - y_\alpha(x^k, \bar{x})\|^2 \ge f(\bar{x}, \bar{x}) + \frac{\alpha}{2}\|\bar{x} - \bar{x}\|^2 + \frac{\alpha+\tau}{2}\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2,$$

hence

$$f(\bar{x}, y_\alpha(x^k, \bar{x})) \ge \frac{\tau}{2}\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2. \tag{22}$$

Denoting $y^k = y_\alpha(x^k, \bar{x})$, we obtain

$$\begin{aligned} \tfrac{\alpha+\tau}{2}\|y^k - \bar{x}\|^2 &\le \tfrac{\alpha}{2}\|x^k - \bar{x}\|^2 + f(x^k, \bar{x}) - f(x^k, y^k) - \tfrac{\alpha}{2}\|y^k - x^k\|^2 \\ &= \tfrac{\alpha}{2}\|x^k - \bar{x}\|^2 + f(x^k, \bar{x}) + f(\bar{x}, x^k) - f(\bar{x}, x^k) - f(x^k, y^k) - \tfrac{\alpha}{2}\|y^k - x^k\|^2 \\ &\le \tfrac{\alpha}{2}\|x^k - \bar{x}\|^2 - \mu\|x^k - \bar{x}\|^2 - f(\bar{x}, y^k) + T_1\|x^k - \bar{x}\|^2 \\ &\qquad + T_2\|y^k - x^k\|^2 - \tfrac{\alpha}{2}\|y^k - x^k\|^2 \\ &\le (\tfrac{\alpha}{2} + T_1 - \mu)\|x^k - \bar{x}\|^2 - \tfrac{\tau}{2}\|y^k - \bar{x}\|^2 + (T_2 - \tfrac{\alpha}{2})\|y^k - x^k\|^2 \\ &\le (\tfrac{\alpha}{2} + T_1 - \mu)\|x^k - \bar{x}\|^2 - \tfrac{\tau}{2}\|y^k - \bar{x}\|^2, \end{aligned}$$

where the first inequality is (21), the second follows from assumptions (M) and (T), the third from (22) and the last one holds since $\alpha \geq 2T_2$. Notice that $\alpha/2 + T_1 - \mu \geq T_2 + T_1 - \mu \geq 0$ by Proposition 2.1 d).

Hence we have

$$\|y^k - \bar{x}\| \leq \sqrt{\frac{\alpha + 2(T_1 - \mu)}{\alpha + 2\tau}}\|x^k - \bar{x}\|. \tag{23}$$

Combining (20) and (23) we obtain

$$\|x^{k+1} - \bar{x}\| \leq [\Lambda(\alpha) + r_2(\alpha)]\|x^k - \bar{x}\|.$$

Since $\Lambda(\alpha) + r_2(\alpha) < 1$, the sequence $\{x^k\}$ linearly converges to $\bar{x}$ as $k \to +\infty$. □

Theorem 3.3. *Suppose that $\bar{x}$ is a solution of (QEP) and assumptions (PM) and (T) are satisfied with $\tau + 2(\rho - T_2) > 0$. If (LB) holds for some $\alpha \geq 2T_1$ and $\Lambda(\alpha) + r_3(\alpha) < 1$, where*

$$r_3(\alpha) := \sqrt{\frac{\alpha}{\alpha + \tau + 2(\rho - T_2)}}, \tag{24}$$

then the sequence $\{x^k\}$ converges to $\bar{x}$ with the linear rate of convergence $\Lambda(\alpha) + r_3(\alpha)$.

Proof. It follows from (20) that

$$\|x^{k+1} - \bar{x}\| \leq \Lambda(\alpha)\|x^k - \bar{x}\| + \|y_\alpha(x^k, \bar{x}) - \bar{x}\|. \tag{25}$$

Moreover, (21) implies

$$\begin{aligned}
(\alpha + \tau)\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 &\leq 2[f(x^k, \bar{x}) - f(x^k, y_\alpha(x^k, \bar{x}))] + \alpha\|x^k - \bar{x}\|^2 - \alpha\|y_\alpha(x^k, \bar{x}) - x^k\|^2 \\
&\leq 2f(y_\alpha(x^k, \bar{x}), \bar{x}) + 2T_1\|y_\alpha(x^k, \bar{x}) - x^k\|^2 + 2T_2\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 \\
&\quad + \alpha\|x^k - \bar{x}\|^2 - \alpha\|y_\alpha(x^k, \bar{x}) - x^k\|^2 \\
&\leq -2\rho\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 + 2T_1\|y_\alpha(x^k, \bar{x}) - x^k\|^2 \\
&\quad + \alpha\|x^k - \bar{x}\|^2 - \alpha\|y_\alpha(x^k, \bar{x}) - x^k\|^2 + 2T_2\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 \\
&\leq 2(T_2 - \rho)\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 + \alpha\|x^k - \bar{x}\|^2
\end{aligned}$$

where the second inequality comes from condition (T), the third from (PM) and the last from the assumption on α. Therefore,

$$[\alpha + \tau + 2(\rho - T_2)]\|y_\alpha(x^k, \bar{x}) - \bar{x}\|^2 \leq \alpha\|x^k - \bar{x}\|^2.$$

The assumption on α implies the thesis. □

The following examples show the independence of the three convergence theorems.

Example 3.1. ***(Theorem 3.1 can be applied but the other two cannot)***
Let $f(x, y) = x^2 - 4xy + 3y^2$ and $C(x) = \{y \in \mathbb{R} : \; y \leq b(x)\}$, where $b : \mathbb{R} \to \mathbb{R}$ is L_b-Lipschitz continuous with $L_b < 1/3$. The convexity assumption holds with $\tau = 6$. Moreover,

assumptions (MG), (LG) and (LB) are satisfied with $\nu = -4$*,* $L = 4$ *and* $\Lambda(\alpha) = L_b$*. Hence, (M), (PM) and (T) hold with* $\mu = \rho = -4$ *and* $T_1 = T_2 = 2$*. The existence of a unique solution and the convergence of the algorithm are guaranteed only by Theorem 3.1. In fact,* $\Lambda(\alpha) + r_1(\alpha) = L_b + (|\alpha| + 4)/(\alpha + 6)$ *holds and its optimal value* $L_b + 2/3 < 1$ *is obtained for* $\alpha = 0$*. On the other hand, Theorem 3.2 and Theorem 3.3 cannot be exploited since* $\Lambda(\alpha) + r_2(\alpha) = L_b + 1 > 1$ *for any* $\alpha \geq 4$ *and* $\tau + 2(\rho - T_2) = -6 < 0$*.*

Example 3.2. ***(Theorem 3.2 can be applied but the other two cannot)***
Let $f(x, y) = |y| - |x| + x(y - x)$ *and* $C(x) = [0, 1] + \beta x$ *where* $\beta \in (1 - \sqrt{2}/2, 1)$*. A solution of (QEP) is* $\bar{x} = 0$ *and* $\tau = 0$ *holds. Assumptions (M) and (T) are satisfied with* $\mu = 1$ *and* $T_1 = T_2 = 1/2$*; hence (PM) is also true with* $\rho = 1$*. Condition (LB) can be verified directly. In fact,*

$$y_\alpha(x, z) = \arg\min\{\alpha y^2/2 + (1 - \alpha)xy + |y| : \ y \in [\beta z, 1 + \beta z]\}$$

holds for any $x, z \in \mathbb{R}$*. Since the above objective function is strongly convex on* $\mathbb{R}$*, there exists a unique minimizer* y_v *on* $\mathbb{R}$*. Moreover, the same function is decreasing for* $y < y_v$ *and increasing for* $y > y_v$*. Therefore, (LB) holds with* $\Lambda(\alpha) = \beta$ *for any* $\alpha > 0$*. Theorem 3.2 guarantees the convergence of the algorithm since* $\Lambda(\alpha) + r_2(\alpha) = \beta + \sqrt{(\alpha - 1)/\alpha}$ *holds and its optimal value* $\beta < 1$ *is obtained for* $\alpha = 1$*. On the other hand, Theorem 3.1 and Theorem 3.3 cannot be exploited since* f *is not continuously differentiable and* $\Lambda(\alpha) + r_3(\alpha) = \beta + \sqrt{\alpha/(\alpha + 1)} \geq \beta + 1/\sqrt{2} > 1$ *for any* $\alpha \geq 1$*.*

Example 3.3. ***(Theorem 3.3 can be applied but the other two cannot)***
Let $f(x, y) = (4 - x)(y - x)$ *and* $C(x) = [0, 1] + \beta \sin^2 x$ *with* $\beta \in (0, 1 - \sqrt{2}/2)$*. A solution of (QEP) is* $\bar{x} = 0$ *and* $\tau = 0$ *holds. Moreover, (T) holds with* $T_1 = T_2 = 1/2$*; (LB) holds with* $\Lambda(\alpha) = \beta$ *for any* $\alpha > 0$ *and* $C(x) \subseteq [0, \beta + 1] \subset [0, 2]$ *for any* $x \in \mathbb{R}$*. The bifunction* f *is* ρ*-pseudomonotone with* $\rho = 1$ *on* $[0, \beta + 1]$*. In fact, for any* $x, y \in [0, \beta + 1]$ *the following implications hold:*

$$(4 - x)(y - x) \geq 0 \Longrightarrow y - x \geq 0 \Longrightarrow (4 - y)(x - y) \leq (3 - \beta)(x - y) \leq 2(x - y) \leq -(x - y)^2.$$

The condition on β *implies that* $\Lambda(\alpha) + r_3(\alpha) < 1$ *for* $\alpha = 1$*, thus the convergence of the algorithm is guaranteed since the assumptions of Theorem 3.3 are satisfied. On the other hand, (MG) and (LG) hold with* $\nu = -1$ *and* $L = 1$*, hence Theorem 3.1 cannot be applied since* $\Lambda(\alpha) + r_1(\alpha) = \beta + (\alpha + 1)/\alpha > 1$ *for any* $\alpha > 0$*. Moreover, (M) holds with* $\mu = -1$*, hence Theorem 3.2 cannot be applied since* $\Lambda(\alpha) + r_2(\alpha) = \beta + \sqrt{(\alpha + 3)/\alpha} > 1$ *for any* $\alpha \geq 1$*.*

Remark 3.3. ***(Comparison between*** r_1 ***and*** r_2***)***
If (MG) and (LG) are satisfied, then Proposition 2.1 guarantees that also (M) and (T) hold with $\mu = \nu$ *and* $T_1 = T_2 = L/2$*. On the other hand, the rate of convergence given by Theorem 3.1 is better than the one given by Theorem 3.2. In fact, for any* $\alpha \geq 2T_2 = L$ *the*

following chain of inequalities holds:

$$\begin{aligned} r_1(\alpha) = \frac{\sqrt{\alpha^2 - 2\alpha\nu + L^2}}{\alpha + \tau} &= \frac{\sqrt{\alpha^2 - 2\alpha\mu + L^2}}{\alpha + \tau} \\ &= \sqrt{\frac{\alpha^2 - 2\alpha\mu + L^2}{\alpha^2 + 2\alpha\tau + \tau^2}} \\ &= \sqrt{\frac{\alpha - 2\mu + L^2/\alpha}{\alpha + 2\tau + \tau^2/\alpha}} \\ &\leq \sqrt{\frac{\alpha - 2\mu + L^2/\alpha}{\alpha + 2\tau}} \\ &\leq \sqrt{\frac{\alpha - 2\mu + L}{\alpha + 2\tau}} \\ &= \sqrt{\frac{\alpha + 2(T_1 - \mu)}{\alpha + 2\tau}} = r_2(\alpha) \end{aligned}$$

Remark 3.4. ***(Comparison between r_1 and r_3)***
Suppose that the assumptions (MG) and (LG) of Theorem 3.1 are satisfied with $\tau+2\nu-L > 0$. Then Proposition 2.1 guarantees that also (PM) and (T) hold with $\rho = \nu$ and $T_1 = T_2 = L/2$. On the other hand, the rate of convergence given by Theorem 3.1 is better than the one given by Theorem 3.3, that is $r_1(\alpha) < r_3(\alpha)$ for any $\alpha \geq 2T_1 = L$. In fact, the inequality

$$r_1(\alpha) = \sqrt{\frac{\alpha - 2\nu + L^2/\alpha}{\alpha + 2\tau + \tau^2/\alpha}} \leq \sqrt{\frac{\alpha - 2\nu + L}{\alpha + 2\tau + \tau^2/\alpha}}$$

holds since $\alpha \geq L$. Moreover,

$$\sqrt{\frac{\alpha - 2\nu + L}{\alpha + 2\tau + \tau^2/\alpha}} < r_3(\alpha) = \sqrt{\frac{\alpha}{\alpha + \tau + 2\nu - L}}$$

if and only if

$$\alpha\tau + \tau^2 + (2\nu - L)(\tau + 2\nu - L) > 0.$$

The latter inequality is true for any $\alpha \geq L$ since

$$\begin{aligned} \alpha\tau + \tau^2 + (2\nu - L)(\tau + 2\nu - L) &> \alpha\tau + \tau^2 - \tau(\tau + 2\nu - L) \\ &= \tau(\alpha - 2\nu + L) \\ &\geq 2\tau(L - \nu) \\ &\geq 0 \end{aligned}$$

where the last inequality follows from Proposition 2.1.

Remark 3.5. ***(Comparison between r_2 and r_3)***
Suppose that the assumptions (M) and (T) of Theorem 3.2 are satisfied with $T_1 = T_2 = T$ and $\tau + 2(\mu - T) > 0$. Then Proposition 2.1 guarantees that also (PM) holds with $\rho = \mu$.

If $\mu \geq 0$, then the rate of convergence given by Theorem 3.2 is better than the one given by Theorem 3.3, that is $r_2(\alpha) \leq r_3(\alpha)$ for any $\alpha \geq 2T$. In fact, if $\tau = 0$ the above inequality holds for any $\alpha \in \mathbb{R}$, while if $\tau > 0$, it holds if and only if

$$\alpha \geq 2(T - \mu)(\tau + 2(\mu - T))/\tau.$$

Since $\tau\mu + 2(T-\mu)^2 \geq 0$, we get $2T \geq 2(T-\mu)(\tau + 2(\mu - T))/\tau$, hence $r_2(\alpha) \leq r_3(\alpha)$ for any $\alpha \geq 2T$.

When $\mu < 0$, the reverse inequality $r_3(\alpha) < r_2(\alpha)$ can hold for some $\alpha \geq 2T$. For instance, if $f(x,y) = (3x+5y)(y-x)$, then (M), (PM) and (T) hold with $\tau = 10$, $\mu = \rho = -2$ and $T_1 = T_2 = 1$. In this case, $r_3(\alpha) < r_2(\alpha)$ for any $\alpha \in [2, 12/5)$.

3.3 Special cases

(EP)

In the special case of a EP, $\Lambda(\alpha) = 0$ holds for any α, hence it is possible to find the values of α which guarantee the convergence of the algorithm under the assumptions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 and the corresponding optimal rates of convergence. In Theorem 3.1, in order to find the values of α guaranteeing the convergence and the optimal rate, three cases on the parameters τ, ν, L have to be analysed (see Table 1). In Theorem 3.2 the function $r_2(\alpha)$ is increasing for $\alpha \geq 2T_2$, thus the optimal rate is

$$r_2(2T_2) = \sqrt{\frac{T_1 + T_2 - \mu}{T_2 + \tau}}.$$

In Theorem 3.3 the function $r_3(\alpha)$ is increasing for $\alpha \geq 2T_1$, thus the optimal rate is

$$r_3(2T_1) = \sqrt{\frac{2T_1}{\tau + 2(\rho + T_1 - T_2)}}.$$

	Assumptions	Convergence	Opt α	Opt rate
Theorem 3.1	(MG), (LG) $\tau \leq L,\ \nu \in (-\tau, L]$	$\alpha > \dfrac{L^2 - \tau^2}{2(\tau+\nu)}$	$\dfrac{L^2 + \tau\nu}{\tau + \nu}$	$\sqrt{\dfrac{L^2 - \nu^2}{(\tau+\nu)^2 + L^2 - \nu^2}}$
	$\tau > L$, $\nu \in [-L^2/\tau, L]$	$\alpha > (L-\tau)/2$	$\dfrac{L^2 + \tau\nu}{\tau + \nu}$	$\sqrt{\dfrac{L^2 - \nu^2}{(\tau+\nu)^2 + L^2 - \nu^2}}$
	$\tau > L$, $\nu \in [-L, -L^2/\tau]$	$\alpha > (L-\tau)/2$	0	L/τ
Theorem 3.2	(M), (T) $\tau + \mu > T_1$	$\alpha \geq 2T_2$	$2T_2$	$\sqrt{\dfrac{T_1 + T_2 - \mu}{T_2 + \tau}}$
Theorem 3.3	(PM), (T) $\tau/2 + \rho > T_2$	$\alpha \geq 2T_1$	$2T_1$	$\sqrt{\dfrac{2T_1}{\tau + 2(\rho + T_1 - T_2)}}$

Table 1: Convergence of the Fixed-point algorithm in the special case of a EP.

(QVI)

Consider the special case of a QVI where the map F is L-Lipschitz continuous and the map C is defined either as in (2), with the assumptions of Proposition 3.1 satisfied, or as in (6), with the assumptions of Proposition 3.2 satisfied (recall that in both cases $\Lambda(\alpha) = \Lambda$ is independent of α).

Theorem 3.1 guarantees that if $\Lambda < 1$ and F is μ-monotone with $\mu > \sqrt{\Lambda(2-\Lambda)}L$ then the algorithm converges for any

$$\alpha \in \left(\frac{\mu - \sqrt{\mu^2 - \Lambda(2-\Lambda)L^2}}{\Lambda(2-\Lambda)}, \frac{\mu + \sqrt{\mu^2 - \Lambda(2-\Lambda)L^2}}{\Lambda(2-\Lambda)} \right),$$

the optimal parameter is $\alpha = L^2/\mu$ and the optimal rate of convergence is $\Lambda + \sqrt{1-(\mu/L)^2}$.

Theorem 3.2 guarantees that if $\Lambda < 1$ and F is μ-monotone with $\mu > (2-(1-\Lambda)^2)L/2$ then the algorithm converges for any

$$\alpha \in \left[L, \frac{2\mu - L}{\Lambda(2-\Lambda)} \right),$$

the optimal parameter is $\alpha = L$ and the optimal rate of convergence is $\Lambda + \sqrt{2(1-\mu/L)}$. Notice that in this case Theorem 3.2 is dominated by Theorem 3.1 since its assumptions are stronger and the optimal rate of convergence is worse, that is the relations $(2-(1-\Lambda)^2)L/2 > \sqrt{\Lambda(2-\Lambda)}L$ and $\sqrt{2(1-\mu/L)} \geq \sqrt{1-(\mu/L)^2}$ hold.

Theorem 3.3 guarantees that if $\Lambda < 1$ and F is ρ-pseudomonotone with $\rho > L/[(2(1-\Lambda)^2]$ then the algorithm converges for any

$$\alpha \in \left[L, \frac{(1-\Lambda)^2(2\rho - L)}{\Lambda(2-\Lambda)} \right),$$

the optimal parameter is $\alpha = L$ and the optimal rate of convergence is $\Lambda + \sqrt{L/(2\rho)}$. Moreover, Theorem 3.3 is not dominated by Theorem 3.1 as Example 3.3 shows.

Furthermore, notice that $L/[(2(1-\Lambda)^2] > \sqrt{\Lambda(2-\Lambda)}L$, hence if F is μ-monotone with $\mu > L/[(2(1-\Lambda)^2]$, then the assumptions of Theorem 3.1 and Theorem 3.3 are satisfied with $\rho = \mu$. In this case, it easy to prove that the optimal rate of convergence of Theorem 3.1 is better than the one of Theorem 3.3.

Remark 3.6. *Recently, in [1] the authors investigated the fixed point algorithm for a QVI where the map F is μ-monotone and L-Lipschitz continuous, the map C is defined as in* (2) *with $s(x) \equiv 1$ and t is L_t-Lipschitz continuous, and prove that the optimal rate of convergence is*

$$q = \sqrt{1 + 3L_t^2 - \frac{4L\mu}{3(L+\mu)^2}}.$$

We notice that the optimal rate given by Theorem 3.1 can be lower than q for some problems. In fact, if we consider a QVI where F and t are such that $\mu = 0.9$, $L = 1$ and $L_t = 0.1$, there $L_t + \sqrt{1-(\mu/L)^2} \simeq 0.536$ while $q \simeq 0.835$.

(VI)

Notice that when QEP is a VI with a L-Lipschitz continuous map F, then Theorem 3.1 guarantees the algorithm is convergent provided that F is ν-monotone with $\nu > 0$ (see [3, Theorem 12.1.2]), while Theorem 3.3 requires F is ρ-pseudomonotone with $\rho > L/2$.

4 Convergence under (EAS)

4.1 Existence of an anchor solution

An alternative assumption to (LB) can be considered, relying on the existence of convex set containing all of the set-images of C and the solvability of the equilibrium problem on this enclosing convex set.

Assumption (EAS): there exists a closed convex set $K \subseteq \mathbb{R}^n$ satisfying

- $C(z) \subseteq K$ for any $z \in K$;
- there exists $x^* \in \bigcap_{z \in K} C(z)$ such that $f(x^*, y) \geq 0$ for any $y \in K$.

(EAS) and (LB) are independent of each other as the following four examples show. The first two provide situations in which (EAS) holds while (LB) does not, while the opposite happens in the last two.

Example 4.1. *Consider the bifunction $f(x, y) = \langle P(x - e), y - x\rangle$, where $P \in \mathbb{R}^{n\times n}$ is positive definite and $e = (1, \ldots, 1) \in \mathbb{R}^n$, and the sets $C(x) = [0, 1]^n + (\sqrt{|x_1 - 1|}, \ldots, \sqrt{|x_n - 1|})$. Then, assumption (EAS) holds with $K = [0, 2]^n$ and $x^* = e$, while (LB) does not hold. In fact, if $n = 2$,*

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and $\alpha > 1$, then for any $x = (1 + \delta, 1 + \delta)$, with $\delta > 0$, there exists a neighborhood U of e such that $y_\alpha(x, z) = (1 + \sqrt{|z_1 - 1|}, 1 + \sqrt{|z_2 - 1|})$ holds for any $z \in U$. Hence, (LB) does not hold since $y_\alpha(x, \cdot)$ is not Lipschitz continuous in U.

Example 4.2. *Consider $f(x, y) = \langle Px, y - x\rangle$, where $P \in \mathbb{R}^{n\times n}$ is positive definite, and*

$$C(x) = \{y \in \mathbb{R}^n : \ y_i \leq b_i(x), \quad i = 1, \ldots, n\},$$

where $b_i(x) = 1 + \sqrt{|x_i|} - \sqrt{|x_i| + 1}$. [this is also a moving set since $C(x) = \mathbb{R}^n_- + b(x)$.] Assumption (EAS) is always true with $K = (-\infty, 1]^n$ and $x^* = 0$, while (LB) does not holds in general. In fact, if $n = 2$,

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and $\alpha > 1$, then for any $x > 0$ there exists a neighborhood U of 0 such that $y_\alpha(x, z) = (b_1(z), b_2(z))$ holds for any $z \in U$. Hence, (LB) does not hold since $y_\alpha(x, \cdot)$ is not Lipschitz continuous in U.

Example 4.3. *Consider $f(x, y) = \langle P(x - a), y - x\rangle$, where $P \in \mathbb{R}^{n\times n}$ is positive definite and $a = e/(2n)$, and $C(x) = s(x)[-1, 1]^n$, where $s(x) = |\cos(\sum_{i=1}^n x_i)|/(2n)$. Assumption (LB) holds since Proposition 3.1 can be applied, while (EAS) does not hold. In fact, suppose by* *contradiction that (EAS) is satisfied, then* $C(0) = [-1/(2n), 1/(2n)]^n \subseteq K$ *since* $0 \in C(x)$ *for any* $x \in \mathbb{R}^n$*. Therefore, the following chain of inclusions holds:*

$$x^* \in \bigcap_{z\in K} C(z) \subseteq \bigcap_{z\in[-1/(2n),1/(2n)]^n} C(z) = \left[-\frac{\cos(1/2)}{2n}, \frac{\cos(1/2)}{2n}\right]^n \subset \left[-\frac{1}{2n}, \frac{1}{2n}\right]^n$$

and hence $x^* \neq a$*. On the other hand,* $f(x^*, y) \geq 0$ *holds for any* $y \in K$ *and* $f(a, y) = 0$ *for any* $y \in K$*, but this is impossible since* P *is positive definite.*

Example 4.4. *Consider* $f(x, y) = \langle Px + Qy, y - x\rangle$*, where* $Q \in \mathbb{R}^{n\times n}$ *is positive semidefinite,* $P - Q \in \mathbb{R}^{n\times n}$ *is positive definite and*

$$C(x) = \{y \in \mathbb{R}^n : \ -y_i \leq \cos(x_i) - 1, \qquad i = 1, \dots, n\}.$$

[this is also a moving set: $C(x) = \mathbb{R}^n_+ + b(x)$*, with* $b_i(x) = 1 - \cos(x_i)$*].*

Assumption (LB) holds since Proposition 3.1 or Proposition 3.2 can be applied, while (EAS) does not hold. In fact, suppose by contradiction that (EAS) is satisfied, then $\mathbb{R}^n_+ \subseteq K$ *since* $(2\pi, \dots, 2\pi) \in C(x)$ *for any* $x \in \mathbb{R}^n$*. Therefore, the following chain of inclusions holds:*

$$x^* \in \bigcap_{z\in K} C(z) \subseteq \bigcap_{z\in\mathbb{R}^n_+} C(z) = [2, +\infty)^n,$$

hence $x^* \neq 0$*. On the other hand,* $f(x^*, y) \geq 0$ *holds for any* $y \in K$ *and* $f(0, y) \geq 0$ *holds for any* $y \in K$*, but this is impossible since* $P - Q$ *is positive definite.*

4.2 Convergence of the algorithm

If the starting point satisfies $x^0 \in K$, then the whole sequence $\{x^k\}$ belongs to K.

Theorem 4.1. *Assume that assumptions (MG), (LG) and (EAS) hold with* $\tau + \nu > 0$*. If* $x^0 \in K$*,* $\alpha > -\tau$ *and* $r_1(\alpha) < 1$*, then the sequence* $\{x^k\}$ *converges to* x^* *with the linear rate of convergence* $r_1(\alpha)$*.*

Proof. Applying the first-order optimality conditions to x^{k+1} and x^*, we have

$$\langle \nabla_2 f(x^k, x^{k+1}) + \alpha(x^{k+1} - x^k), z - x^{k+1}\rangle \geq 0, \qquad \forall\ z \in C(x^k), \tag{26}$$

$$\langle \nabla_2 f(x^*, x^*) + \alpha(x^* - x^*), z - x^*\rangle \geq 0, \qquad \forall\ z \in K. \tag{27}$$

Setting $z = x^*$ in (26) and $z = x^{k+1}$ in (27) and summing the two inequalities, we get

$$\langle \nabla_2 f(x^k, x^{k+1}) - \nabla_2 f(x^*, x^*) + \alpha(x^{k+1} - x^* + x^* - x^k), x^* - x^{k+1}\rangle \geq 0,$$

hence

$$\alpha\|x^{k+1} - x^*\|^2 \leq \langle \nabla_2 f(x^k, x^{k+1}) - \nabla_2 f(x^*, x^*) + \alpha(x^* - x^k), x^* - x^{k+1}\rangle. \tag{28}$$

Moreover, the τ-convexity of $f(x^k, \cdot)$ implies the map $\nabla_2 f(x^k, \cdot)$ is τ-monotone for any k, thus

$$\tau\|x^{k+1} - x^*\|^2 \leq \langle \nabla_2 f(x^k, x^*) - \nabla_2 f(x^k, x^{k+1}), x^* - x^{k+1}\rangle. \tag{29}$$

Summing (28) and (29), we get

$$\begin{aligned}(\alpha+\tau)\|x^{k+1}-x^*\|^2 &\le \langle \nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*) + \alpha(x^*-x^k), x^*-x^{k+1}\rangle \\ &\le \|\nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*) + \alpha(x^*-x^k)\|\|x^{k+1}-x^*\|\end{aligned}$$

that is

$$(\alpha+\tau)\|x^{k+1}-x^*\| \le \|\nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*) + \alpha(x^*-x^k)\|. \tag{30}$$

Therefore, we have

$$\begin{aligned}(\alpha+\tau)^2\|x^{k+1}-x^*\|^2 &\le \|\nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*) + \alpha(x^*-x^k)\|^2 \\ &= \|\nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*)\|^2 + \alpha^2\|x^*-x^k\|^2 \\ &\quad + 2\alpha\langle \nabla_2 f(x^k,x^*) - \nabla_2 f(x^*,x^*), x^*-x^k\rangle.\end{aligned}$$

Arguing as in the proof of Theorem 3.1, we get the thesis. □

Theorem 4.2. *Suppose that assumptions (M), (T) and (EAS) hold. If $\tau+\mu > T_1$, $x^0 \in K$ and $\alpha \ge 2T_2$, then the sequence $\{x^k\}$ converges to x^* with the linear rate of convergence $r_2(\alpha)$.*

Proof. Applying Lemma 3.2 with $h = f(x^k,\cdot)$, $\xi = x^k$, $X = C(x^k)$, $x^+ = x^{k+1}$ and $x = x^*$, we get

$$f(x^k,x^*) + \frac{\alpha}{2}\|x^k-x^*\|^2 \ge f(x^k,x^{k+1}) + \frac{\alpha}{2}\|x^{k+1}-x^k\|^2 + \frac{\alpha+\tau}{2}\|x^{k+1}-x^*\|^2. \tag{31}$$

On the other hand, applying Lemma 3.2 with $h = f(x^*,\cdot)$, $\xi = x^*$, $X = K$, $x^+ = x^*$ and $x = x^{k+1} \in C(x^k) \subseteq K$, and exploiting that x^* solves (QEP), we get

$$f(x^*,x^{k+1}) \ge \frac{\tau}{2}\|x^{k+1}-x^*\|^2, \tag{32}$$

Therefore, we obtain

$$\begin{aligned}\tfrac{\alpha+\tau}{2}\|x^{k+1}-x^*\|^2 &\le \tfrac{\alpha}{2}\|x^k-x^*\|^2 + f(x^k,x^*) - f(x^k,x^{k+1}) - \tfrac{\alpha}{2}\|x^{k+1}-x^k\|^2 \\ &= \tfrac{\alpha}{2}\|x^k-x^*\|^2 + f(x^k,x^*) + f(x^*,x^k) - f(x^*,x^k) - f(x^k,x^{k+1}) \\ &\quad -\tfrac{\alpha}{2}\|x^{k+1}-x^k\|^2 \\ &\le \tfrac{\alpha}{2}\|x^k-x^*\|^2 - \mu\|x^k-x^*\|^2 - f(x^*,x^{k+1}) + T_1\|x^k-x^*\|^2 \\ &\quad +T_2\|x^{k+1}-x^k\|^2 - \tfrac{\alpha}{2}\|x^{k+1}-x^k\|^2 \\ &\le (\tfrac{\alpha}{2}+T_1-\mu)\|x^k-x^*\|^2 - \tfrac{\tau}{2}\|x^{k+1}-x^*\|^2 + (T_2-\tfrac{\alpha}{2})\|x^{k+1}-x^k\|^2 \\ &\le (\tfrac{\alpha}{2}+T_1-\mu)\|x^k-x^*\|^2 - \tfrac{\tau}{2}\|x^{k+1}-x^*\|^2,\end{aligned}$$

where the first inequality is (31), the second follows from assumptions (M) and (T), the third from (32) and the last one holds since $\alpha \geq 2T_2$. Notice that $\alpha/2 + T_1 - \mu \geq T_2 + T_1 - \mu \geq 0$ by Proposition 2.1 d).

Hence we have

$$\|x^{k+1} - x^*\| \leq \sqrt{\frac{\alpha + 2(T_1 - \mu)}{\alpha + 2\tau}} \|x^k - x^*\|. \tag{33}$$

Since $\tau + \mu > T_1$ by assumption, the sequence $\{x^k\}$ converges to x^* when $k \to +\infty$. □

Theorem 4.3. *Suppose that assumptions (PM), (T) and (EAS) hold. If* $\tau + 2\mu > 2T_2$, $x^0 \in K$ *and* $\alpha \geq 2T_1$, *then the sequence* $\{x^k\}$ *converges to* x^* *with the linear rate of convergence* $r_3(\alpha)$.

Proof. It follows from (31) that

$$\begin{aligned}
(\alpha + \tau)\|x^{k+1} - x^*\|^2 &\leq 2[f(x^k, x^*) - f(x^k, x^{k+1})] + \alpha\|x^k - x^*\|^2 - \alpha\|x^{k+1} - x^k\|^2 \\
&\leq 2f(x^{k+1}, x^*) + 2T_1\|x^{k+1} - x^k\|^2 + 2T_2\|x^{k+1} - x^*\|^2 \\
&\quad + \alpha\|x^k - x^*\|^2 - \alpha\|x^{k+1} - x^k\|^2 \\
&\leq -2\mu\|x^{k+1} - x^*\|^2 + 2T_1\|x^{k+1} - x^k\|^2 + 2T_2\|x^{k+1} - x^*\|^2 \\
&\quad + \alpha\|x^k - x^*\|^2 - \alpha\|x^{k+1} - x^k\|^2 \\
&\leq 2(T_2 - \mu)\|x^{k+1} - x^*\|^2 + \alpha\|x^k - x^*\|^2
\end{aligned}$$

where the second inequality comes from condition (T), the third from (PM) and the last from the assumption on α. Therefore,

$$[\alpha + \tau + 2(\mu - T_2)]\|x^{k+1} - x^*\|^2 \leq \alpha\|x^k - x^*\|^2.$$

The assumption on α implies the thesis. □

Remark 4.1. *In the particular case (EP) the value of* α *guaranteing convergence and the optimal rates ottimi are again by Table 1.*

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